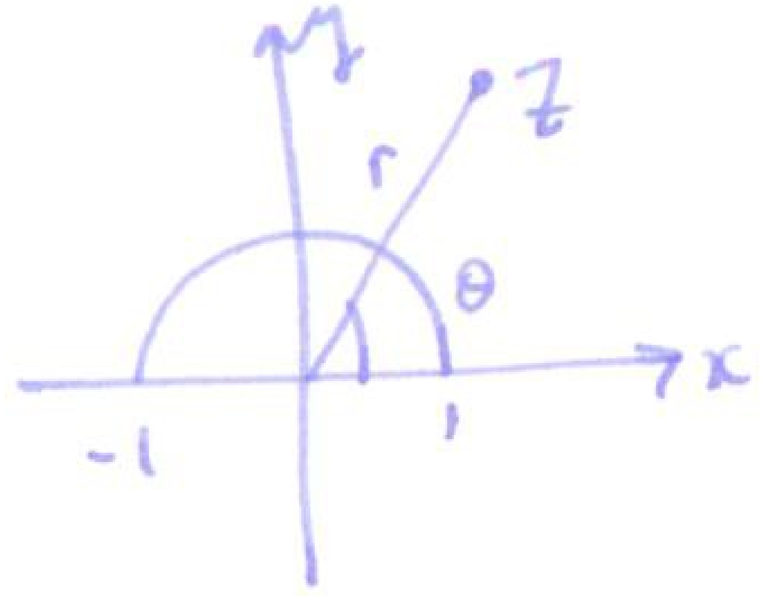
"reflection" in unit circle $\longleftrightarrow$ inversion in circle.



fact: this is given by $z \mapsto \frac{1}{\bar{z}}$

note $z = re^{i\theta}$

$$\frac{1}{\bar{z}} = \frac{1}{re^{-i\theta}} = \frac{1}{r}e^{i\theta}$$

isometries of $\mathbb{H}^2$ / maps of upper half space to itself taking "lines" to "lines"

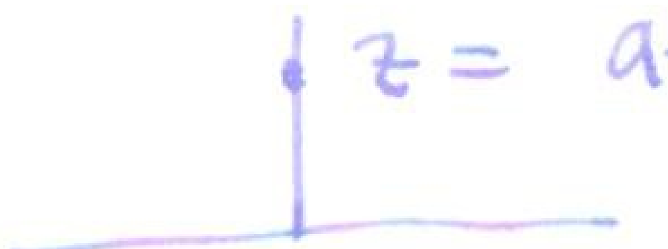
horizontal translations: $z \mapsto z + a \quad a \in \mathbb{R}$.

dilations: $z \mapsto kz \qquad k \in \mathbb{R}_{>0}$

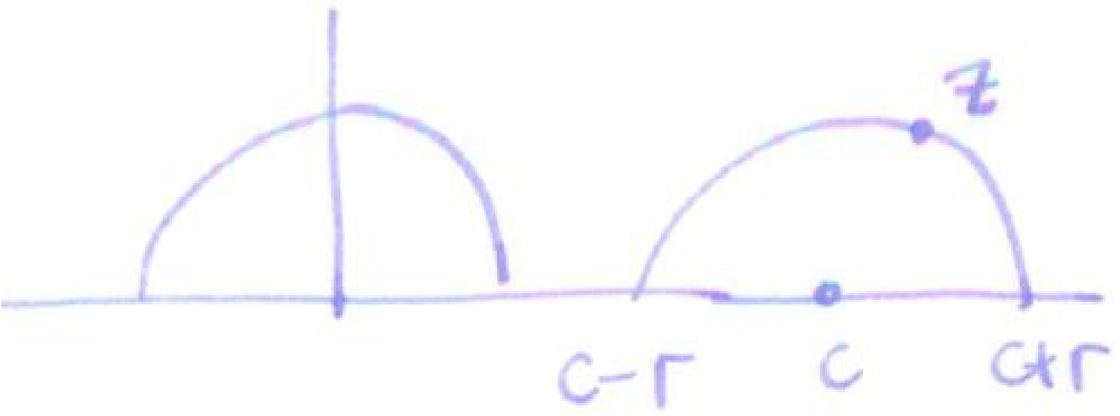
reflection in y-axis: $z \mapsto -\bar{z}$

reflection in unit circle: $z \mapsto \frac{1}{\bar{z}}$.

Equations of non-Euclidean lines



- $z + \bar{z} = 2a$. (*)



- $|z - c| = r$

$$\Leftrightarrow |z-c|^2 = r^2$$

$$\| \quad (z-c)\overline{(z-c)} = r^2$$

$$(z-c)(\bar{z}-\bar{c}) = r^2 \qquad c = \bar{c}! \ (c \text{ real})$$

$$z\bar{z} - c\bar{z} - \bar{c}z + c\bar{c} = r^2$$

$$z\bar{z} - c(z+\bar{z}) + c^2 - r^2 = 0 \quad (**)$$

(*), (**) both of the form $Az\bar{z} + B(z+\bar{z}) + C = 0$ (***) for some $A, B, C \in \mathbb{R}$.

if A, B not both zero, then $(**)$ reduces to one of $(*)$ or $(**)$:

if $A=0$ then: $z+\bar{z}+C/B=0$ $(*)$ $(a=-C/B)$

if $A\neq 0$ then: $z\bar{z}+(z+\bar{z})B/A+C/A=0$ $(**)$ $(c=-B/A \quad c^2-r^2=C/A.)$

(no sol^{ns} if $r^2=B^2/c^2-C/A<0$).

<u>summary</u> every "line" in upper half space model

satisfies: $Az\bar{z}+B(z+\bar{z})+C=0$ for some $A,B,C\in\mathbb{R}$

§8.3 Reflections and Möbius transformations

<u>Def^n</u> the Möbius group is the group of all transformations of the upper half space of the form $z\mapsto\dfrac{az+b}{cz+d}$ or $\dfrac{a\bar{z}+b}{c\bar{z}+d}$ $\quad a,b,c,d\in\mathbb{R}.$

$z: ad-bc>0$

$\bar{z}: ad-bc<0$.

<u>Observations</u> . our isometries are of this form:

Euclidean translation $z\mapsto z+a.$ $(a\in\mathbb{R})$

Euclidean dilation $z\mapsto az$ $(a\in\mathbb{R}_+)$

reflection in y-axis $z\mapsto-\bar{z}$

inversion in circle $z\mapsto-\dfrac{1}{\bar{z}}$

. every element of Möbius group is a product of special isometries.

$z\mapsto cz \qquad z\mapsto z+d \qquad z\mapsto\dfrac{1}{\bar{z}} \qquad z\mapsto\left(b-\dfrac{ad}{c}\right)z$

$z\mapsto z+\dfrac{a}{c}$. gives $z\mapsto\dfrac{a\bar{z}+b}{c\bar{z}+d}$

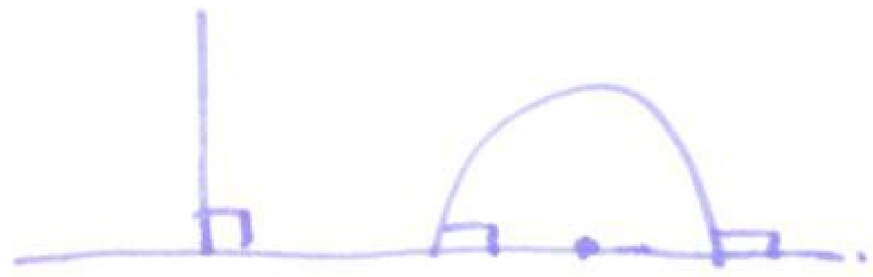
summary . special isometries ~~generate~~ are elements of the Möbius group

. all elements of the Möbius group are products of special isometries.

§8.4 Preserving hyperbolic lines

Thm Every Möbius transformation ~~preserves~~ maps hyperbolic lines to hyperbolic lines.

Proof every element of the Möbius group is a product of the special isometries, so it suffices to check those.



$z \mapsto z+a$ ✓.

$z \mapsto az$ ✓

$z \mapsto -\bar{z}$: reflection in y-axis ✓.

$z \mapsto -\frac{1}{\bar{z}}$

Hyperbolic lines correspond to equations of the form $Az\bar{z} + B(z+\bar{z}) + C = 0$, $A, B, C \in \mathbb{R}$

send $z \xrightarrow{w=} \frac{1}{\bar{z}}$: $\quad A\left(\frac{1}{w}\frac{1}{\bar{w}}\right) + B\left(\frac{1}{\bar{w}} + \frac{1}{w}\right) + C = 0$.

(i.e. $z = \frac{1}{\bar{w}}$).

multiply by $w\bar{w}$:

$$A + B(w+\bar{w}) + Cw\bar{w} = 0$$

this is the equation of a hyperbolic line. $\square$.

§8.5 Preserving angles

angle between curves = angle between tangents

Thm The Möbius group preserves angles.

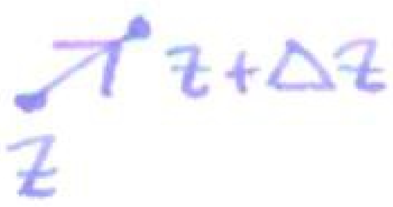
Proof every Möbius group element is a product of special isometries, so it suffices to check those.

$z \mapsto z+a$ Euclidean translation: preserves angles

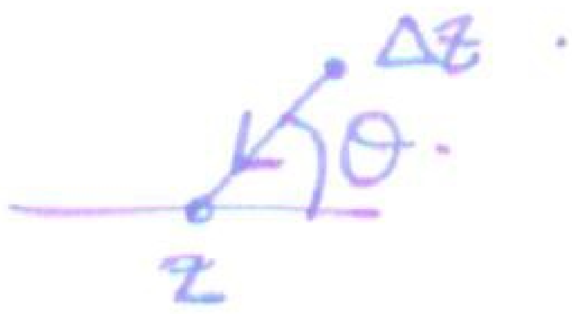
$z \mapsto az$ Euclidean dilation: preserves angles.

$z \mapsto -\bar{z}$ Euclidean reflection: preserves angles.

$z \mapsto -\frac{1}{z}$ directions parameterized by $z, z+\Delta z$



where $\Delta z = \epsilon(\cos\theta + i\sin\theta)$



note: as $\epsilon \to 0$ $\Delta z \to z$ from direction θ

$z \mapsto -\frac{1}{z}$ sends: $z \mapsto -\frac{1}{z}$

$z+\Delta z \mapsto -\frac{1}{z+\Delta z}$

difference:

$$-\frac{1}{z} + \frac{1}{z+\Delta z} = \frac{-z-\Delta z+z}{z(z+\Delta z)} = \frac{-\Delta z}{z(z+\Delta z)}$$

$$= \frac{-\epsilon(\cos\theta + i\sin\theta)}{z(z+\epsilon(\cos\theta + i\sin\theta))} \underset{\epsilon \to 0}{\approx} \cancel{\frac{-\epsilon}{z}} -\epsilon \cdot \underbrace{\frac{1}{z^2}}_{\text{some constant}} (\cos\theta + i\sin\theta)$$