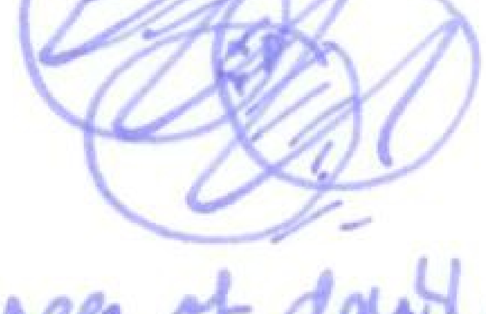
area of unit sphere $= 4\pi$

	angles	area
 this triangle has area $\frac{\pi}{2}$ $\left(= \frac{4\pi}{8}\right)$	$\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$	$\frac{\pi}{2}$.
	$\frac{\pi}{6}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$	$\frac{\pi}{6}$
	π $\frac{\pi}{2}$ $\frac{\pi}{2}$	π
	$\frac{\pi}{4}$ $\frac{\pi}{4}$ π	$\frac{\pi}{2}$
	π π π	2π

<u>Thm</u> (Girard) area of triangle on S^2 = $\alpha+\beta+\gamma-\pi$.

<u>Proof</u>

double lune: area is $4\pi\frac{\alpha}{\pi} = 4\alpha$



3 double lunes.

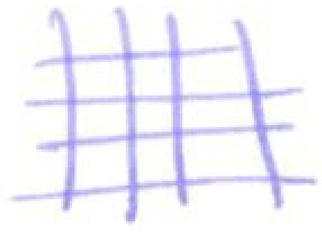
angles	areas
α	4α
β	4β
γ	4γ

area of double lunes = area of sphere + 2 area of triangle + 2 area of antipodal triangle

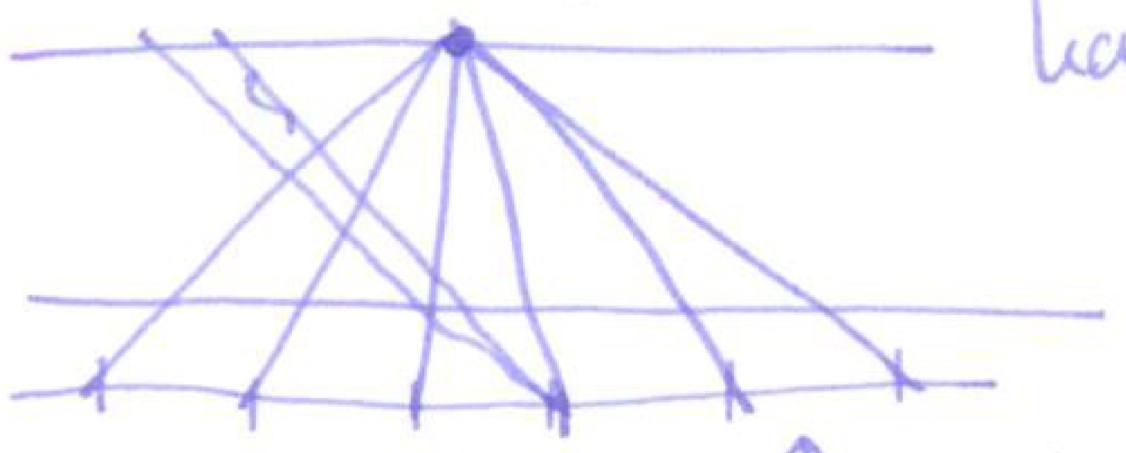
$4\alpha+4\beta+4\gamma = 4\pi + 4A$.

$A = \alpha+\beta+\gamma-\pi$. $\square$

§5.1 Perspective drawing

Q: how do we draw a picture of a tiled floor? 

← point at ∞ (vanishing point)



horizon (line of infinity)

← draw another horizontal line.

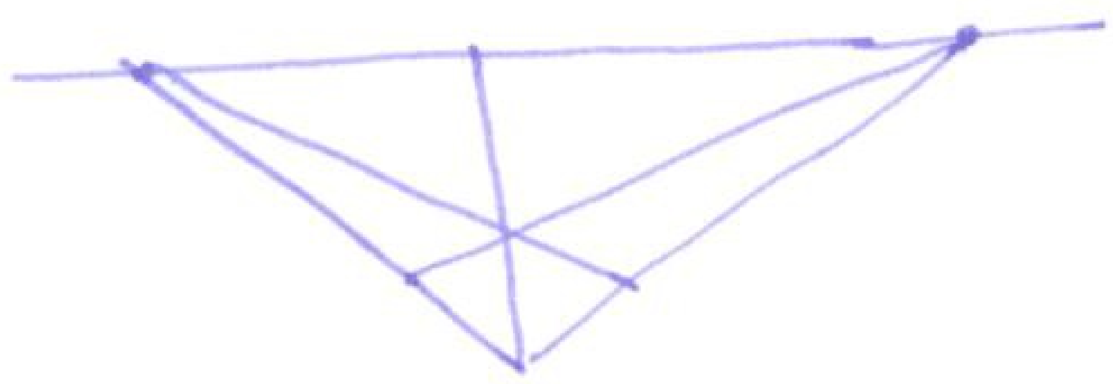
↑ equally spaced points

Q: ~~how~~ where do we draw the remaining horizontal lines?

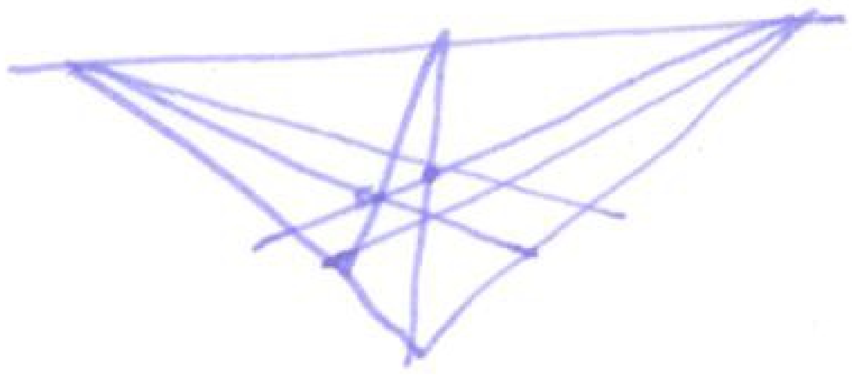
A: draw a diagonal across one tile, now draw horizontals at the intersections.

§5.2 Drawing with the straightedge only

start with one tile and the horizon.



draw diagonal – this gives point at infinity for all diagonals parallel to this one.



so can draw in adjacent parallels.

this gives a corner of the adjacent tile

this gives another adjacent tile corner, and so on

§5.3 Projective plane axioms and their models

In any view (perspective drawing) of the plane:

- straight lines are straight.
- intersections remain intersections.
- parallel lines either remain parallel, or meet at the horizon.

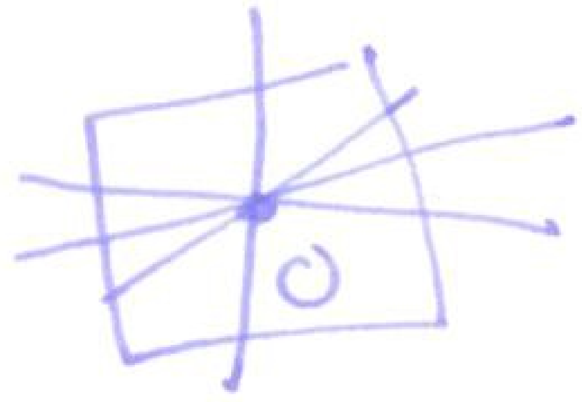
note: parallel lines always meet at the horizon if you look in the right direction...

Projective plane axioms

1. any two "points" are contained in a unique "line"
2. any two lines contain a unique "point"
3. there are four "points" no three of which lie on a "line".

Q: does anything satisfy these axioms?

The real projective plane $\mathbb{RP}^2$.

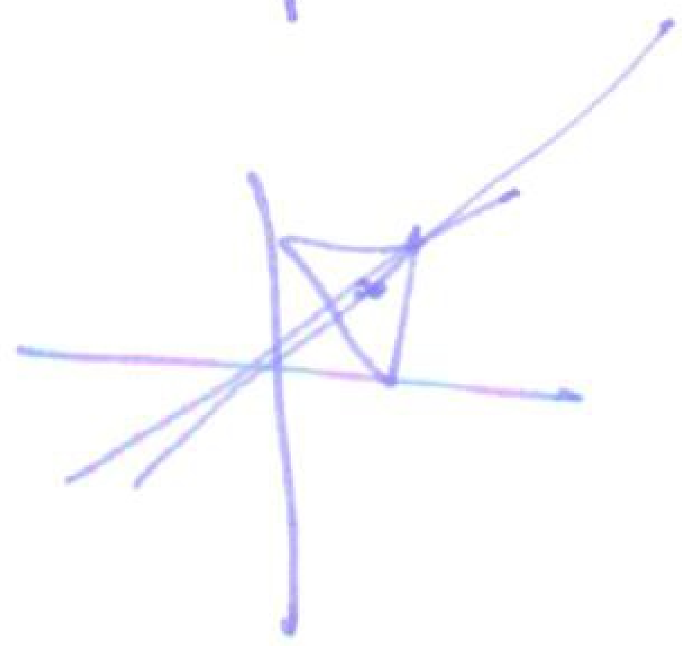


"points" $\leftrightarrow$ lines through 0 in $\mathbb{R}^3$

"lines" $\leftrightarrow$ planes through 0 in $\mathbb{R}^3$

check: (1) any two distinct lines determine a unique plane.

(2) any two distinct planes intersect in a unique line.



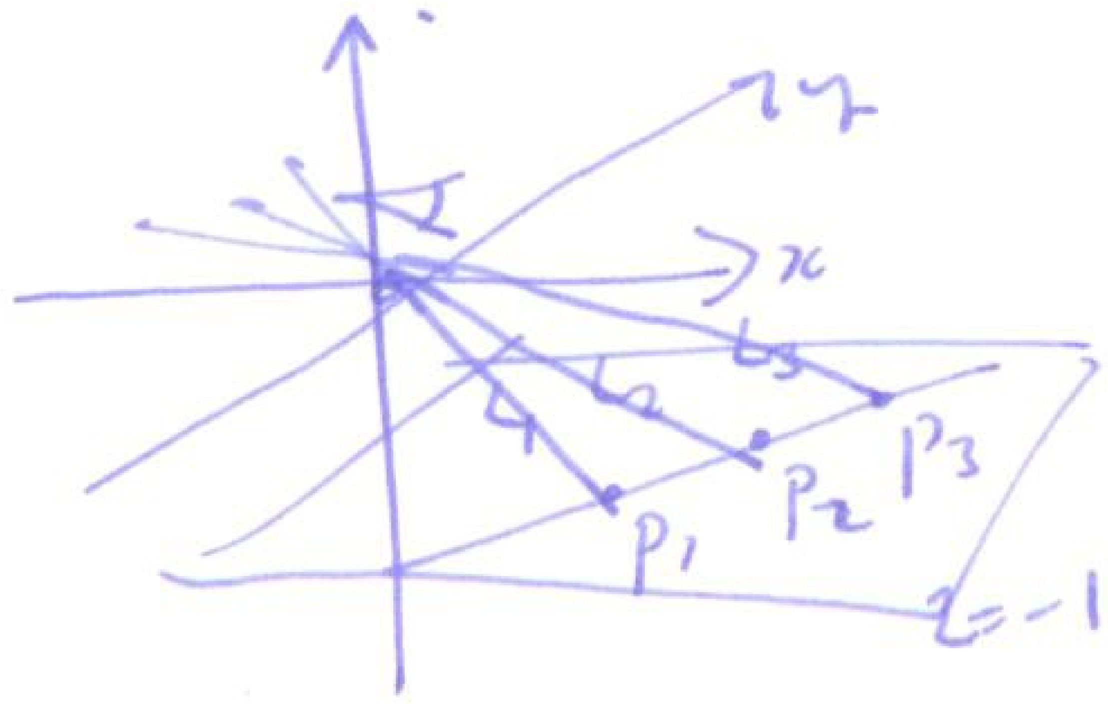
(3) lines through $\underbrace{(1,0,0), (0,1,0), (0,0,1)}_{\text{axes}}$ $(1,1,1)$.

every plane through 0 has equation $ax+by+cz=0$

if $(1,0,0)$ and $(0,1,0)$ lie on the plane then $a=0$, $b=0$

$\Rightarrow$ plane is $z=0$, does not contain $(0,0,1)$ $(1,1,1)$.

other case similarly.



points P_1, P_2, P_3 in the plane connected to 0 by lines $L_1\ L_2\ L_3$.

horizontal lines in $\mathbb{R}^3$ don't correspond to points in $z=-1$, correspond to points "at infinity" (on the horizon).

§6 Projective planes

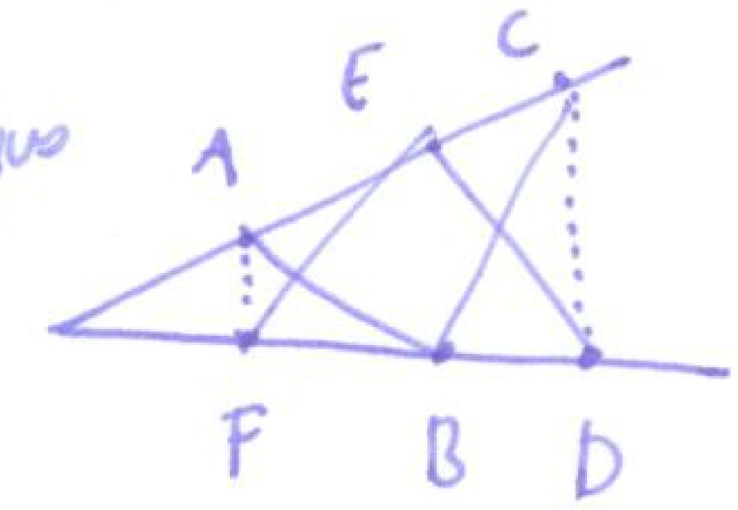
we can build coordinates using geometry.

need: 3 axioms for projective plane + Pappus $\Longleftrightarrow$ $ab = ba$

Desargues $\Longleftrightarrow$ $a(bc) = (ab)c$.

§6.1 Pappus and Desargues

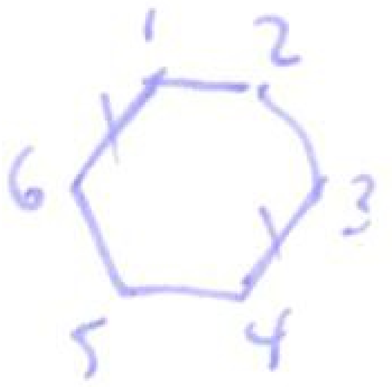
recall: Pappus

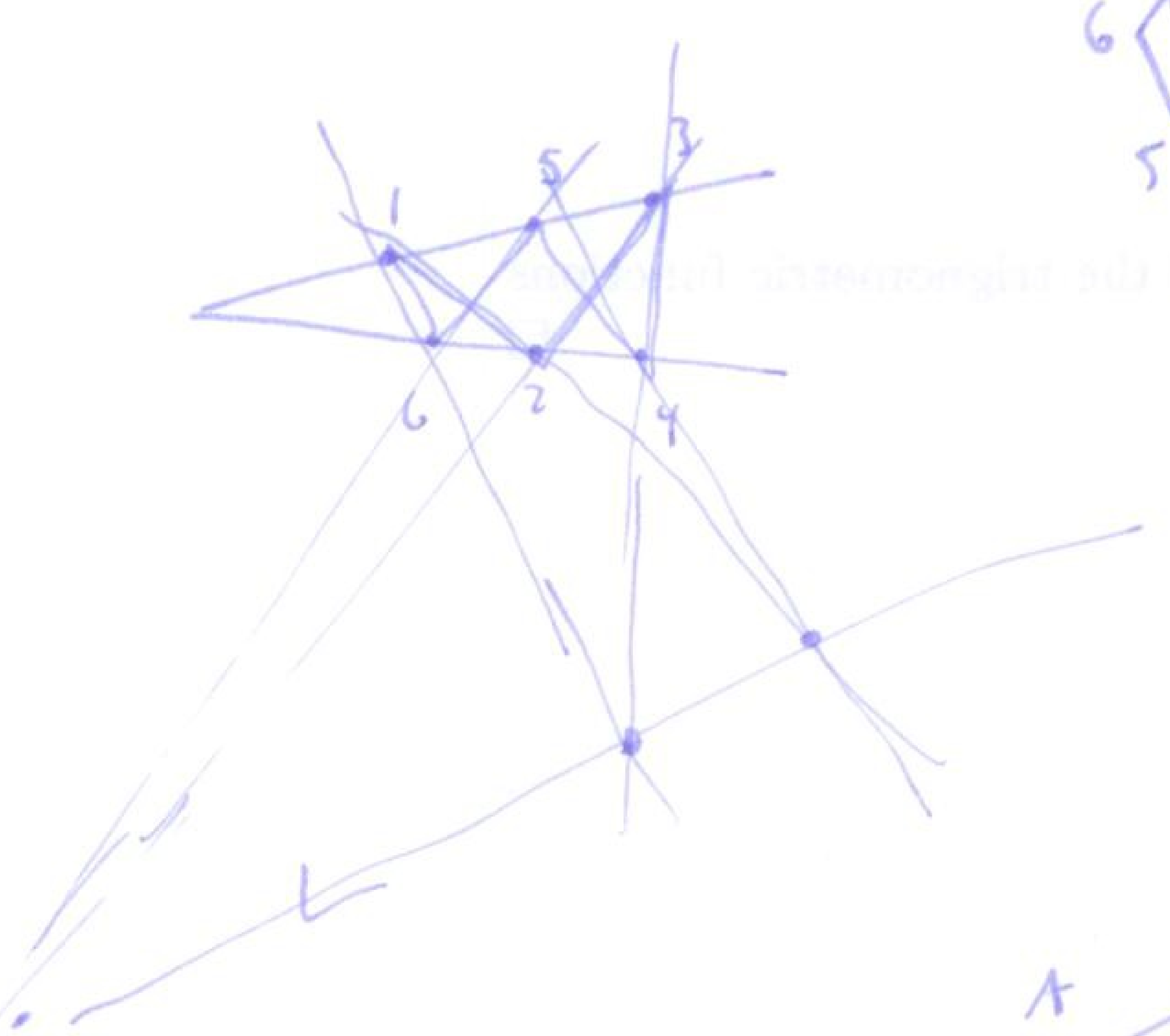


$AB \parallel ED,\ FE \parallel BC \Rightarrow AF \parallel CD$

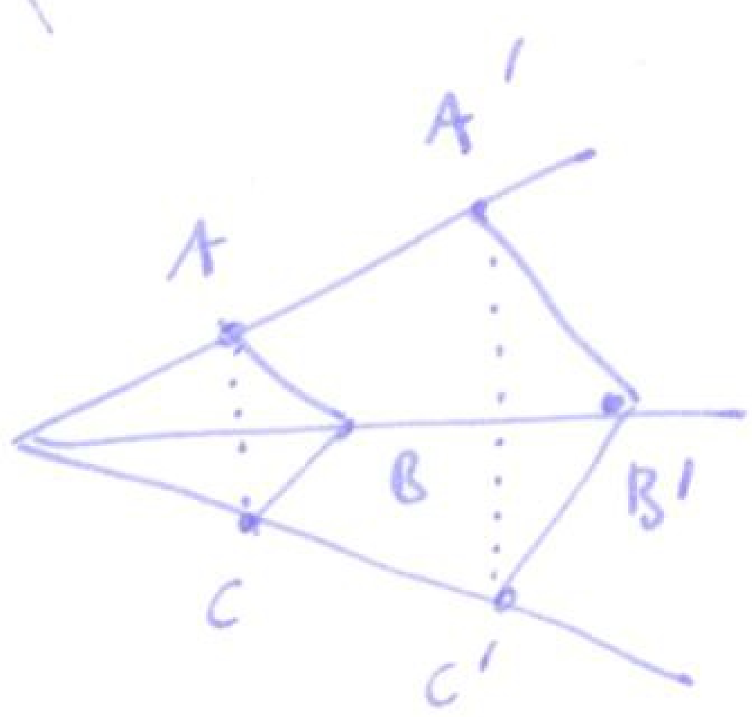
in projective geometry: parallel $\Longleftrightarrow$ meet on horizon (some other line)

Projective Pappus Thm: Six points, lying alternately on two straight lines form a hexagon whose three pairs of opposite sides meet on a common line.





recall: Desargues



$AB \parallel A'B',\ BC \parallel B'C' \Rightarrow AC \parallel A'C'$

$\leftarrow$ we say these two triangles are in perspective form (lines through matching vertices meet in a point).

projective geometry: parallel $\Longleftrightarrow$ meet on horizon (some other line)

Projective Desargues Thm: If two triangles are in perspective from a point, then their pairs of corresponding sides meet in a line.

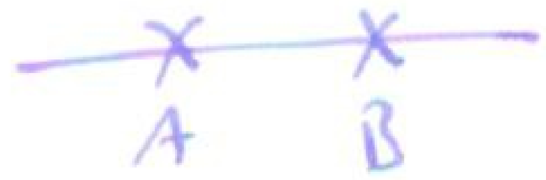
special case (Little Desargues Thm) If two triangles are in perspective from a point P, and if two pairs of corresponding sides meet on a line L through P, then the third pair of corresponding sides also meets on L.

note: there are projective planes that do not satisfy Pappus + Desargues. so we take projective axioms + Pappus + Desargues as axioms, and call these Pappian planes. (e.g. $\mathbb{RP}^2$).

Pappian planes are the projective planes with coordinates.

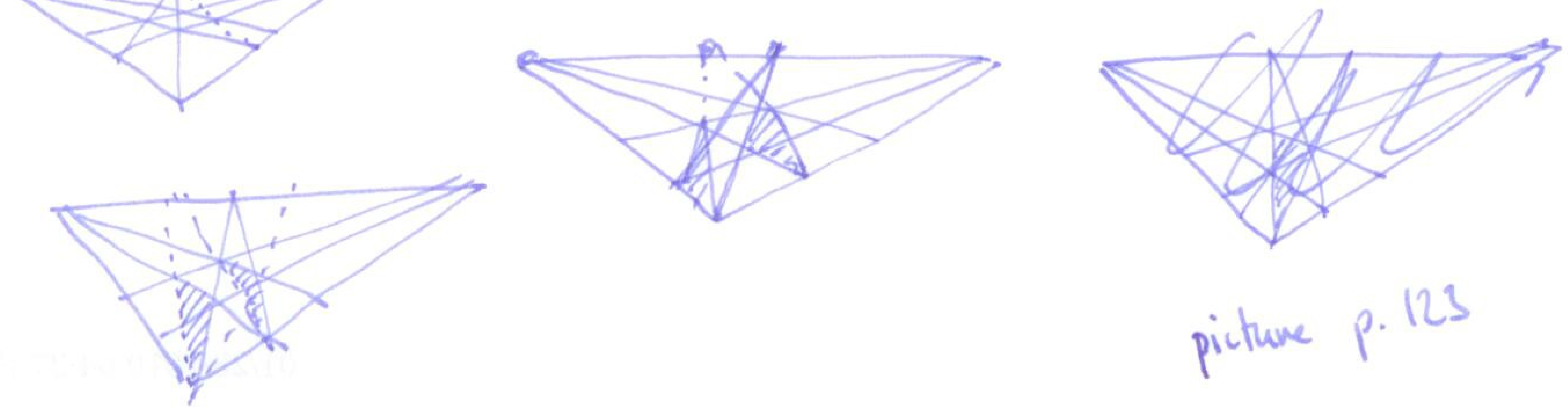
§6.2 Coincidences

two points A, B determine a line
in general a 3rd point C does not lie on the line. If it does we say they are colinear, or coincident.



Example constructing of perspective tiling by straightedge.

coincident point. lines are coincident by little Desargues.



picture p. 123