① ①

$$f:(x,y)\mapsto(cx-sy,\ sx+cy) \qquad c=\cos\left(-\tfrac{\pi}{2}\right)=0$$

$$s=\sin\left(-\tfrac{\pi}{2}\right)=-1$$

$$(y,\ -x)$$

$$g:(x,y)\mapsto(x-2,\ y)\mapsto(y,\ -x+2)\mapsto(y+2,\ -x+2)$$

$$g\circ f:(x,y)\mapsto(y,\ -x)\mapsto(-x+2,\ -y+2)$$

fixed point: $(x,y)=(-x+2,\ -y+2) \Rightarrow (x,y)=(1,1)$

rotation by π about $(1,1)$.

②

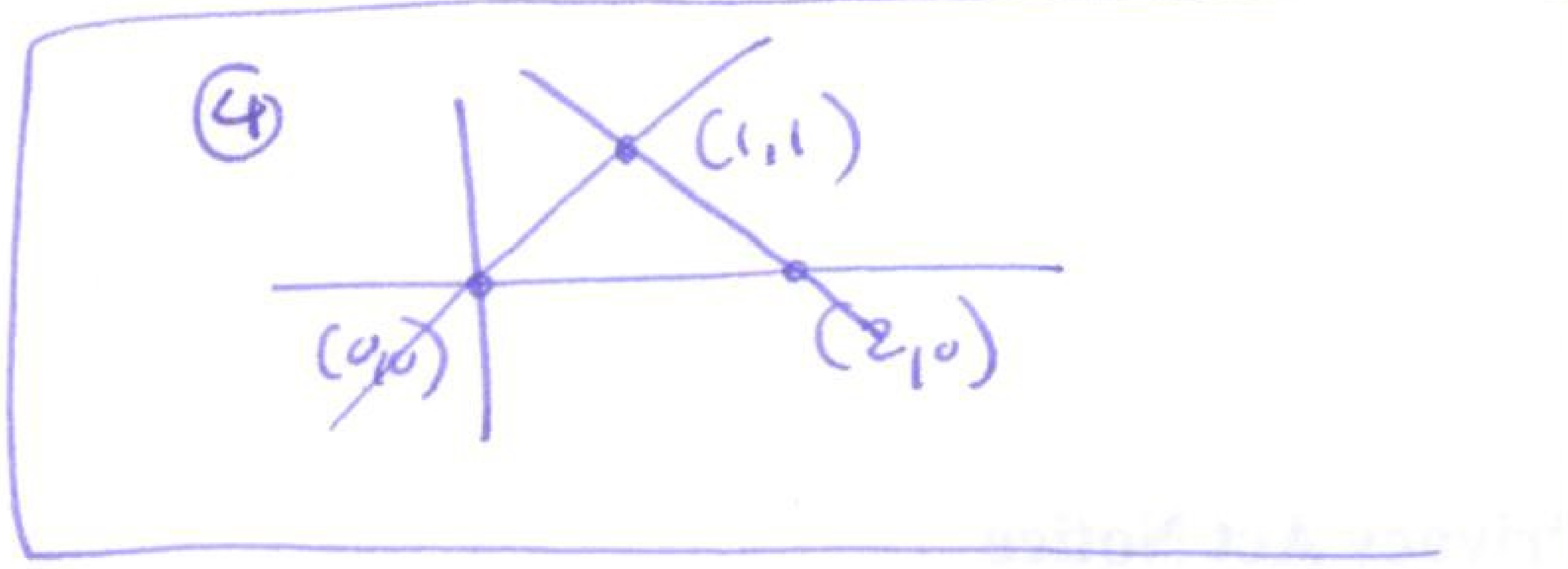
$$f:\begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}c&-s\\s&c\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}y\\-x\end{bmatrix} \qquad \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}$$

$$\|$$

$$g:\begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}x-2\\y\end{bmatrix}\mapsto\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}x-2\\y\end{bmatrix}=\begin{bmatrix}y\\-x+2\end{bmatrix}\mapsto\begin{bmatrix}y+2\\-x+2\end{bmatrix}.$$

$$g\circ f:\begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}y\\-x\end{bmatrix}\mapsto\begin{bmatrix}-x+2\\-y+2\end{bmatrix}$$

④



③ $f: z\mapsto e^{-i\pi/2}z$

$$g: z\mapsto z-2\mapsto e^{-i\pi/2}(z-2)\mapsto e^{-i\pi/2}(z-2)+2=e^{-i\pi/2}z-2e^{-i\pi/2}+2.$$

$$g\circ f: z\mapsto e^{-i\pi/2}z\mapsto e^{-i\pi/2}\left(e^{-i\pi/2}z\right)-2e^{-i\pi/2}+2=e^{-i\pi}z+2-2e^{-i\pi/2}$$

fixed point: solve $z=-z+2-2e^{-i\pi/2}$ $\qquad z=1+i$

Summary

function notation: $(x,y) \mapsto (f(x,y), g(x,y))$ very general — any function $\mathbb{R}^2 \to \mathbb{R}^2$.

matrix notation / linear maps $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto A\underline{x} + \underline{b}$

recall: $\det(A) = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

if $\det(A) \neq 0$ then $A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

note: not all linear maps are isometries eg $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ scales $\times 2$ in x direction.

$\det(A) = 1 \not\Rightarrow$ isometry! e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$.

det means area of unit square.

Q: when is $\underline{x} \mapsto A\underline{x}$ an isometry?

recall $|\underline{x}|^2 = \underline{x} \cdot \underline{x} = \underline{x}^T \underline{x}$ in matrix notation

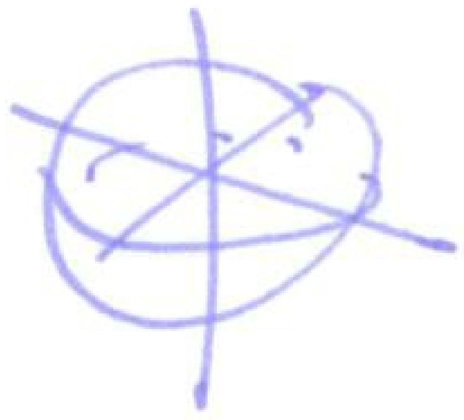
$\underline{x} \cdot \underline{y} = \underline{x}^T \underline{y} = (A\underline{x})^T (A\underline{y}) = \underline{x}^T A^T A \underline{y}$ for all $\underline{x}, \underline{y}$

$\Rightarrow A^T A = I$.

complex notation $z \mapsto e^{i\theta} z + a$ $\qquad e^{i\theta} = \cos\theta + i\sin\theta$.

§7.4 Spherical geometry

sphere $S^2 \subset \mathbb{R}^3$ $\{(x,y,z) \mid x^2+y^2+z^2=1\}$



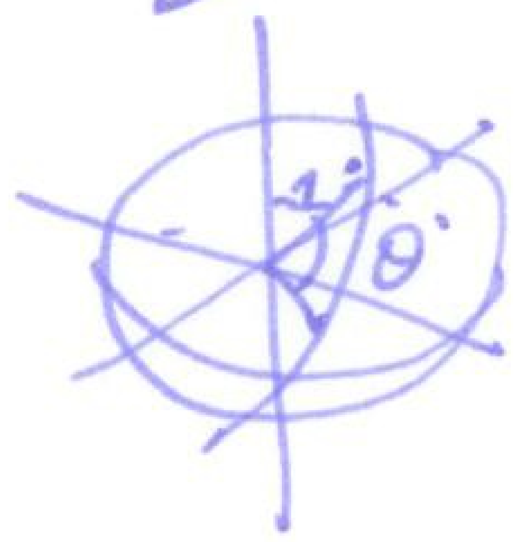
locally two dimensional

points $\leftrightarrow$ rays through $O \cap S^2$

lines $\leftrightarrow$ planes through $O \cap S^2$. [great circles minimize distance $\rightarrow$ use calculus].

circles $\leftrightarrow$ planes $\cap S^2$

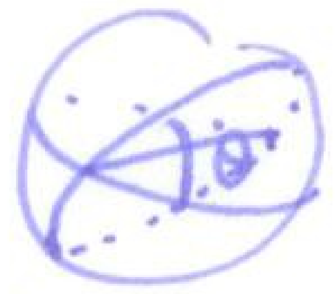
great circle distance $\leftrightarrow$ angle



Isometries of S^2

reflections in lines $\leftrightarrow$ reflections in planes through O
(great circles)

reflection in one line then another:



gives a rotation by 2θ about line of intersection.

reflection in 3 planes gives new isometry, eg. antipodal map

$(x,y,z) \mapsto (-x,-y,-z)$

$$(x,y,z) \xmapsto[\text{yz plane}]{\text{reflect in}} (-x,y,z) \xmapsto[\text{xz plane}]{\text{reflect in}} (-x,-y,z) \xmapsto[\text{(x,y) plane}]{\text{reflect in}} (-x,-y,z)$$

Thm every isometry of S^2 is a product of at most 3 reflections.

Remark: spherical geometry is non-Euclidean. No parallel lines!