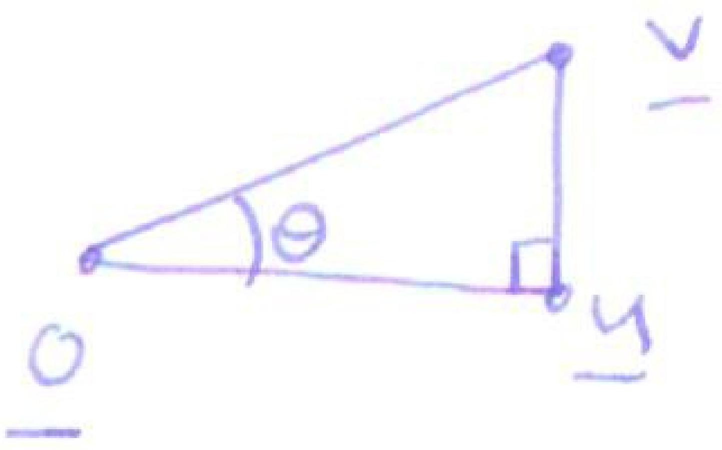
$(\underline{u}-\underline{v}).\underline{w} = 0$ as required $\square$.

§4.5 Inner product and cosine.



$\perp : (\underline{v}-\underline{u}).\underline{u} = 0$

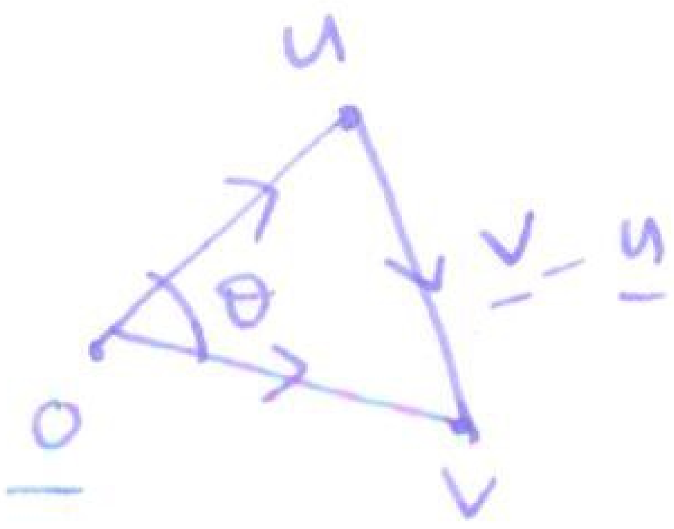
define $\cos\theta = \dfrac{|\underline{u}|}{|\underline{v}|}$

Inner product formula $\quad \underline{u}.\underline{v} = |\underline{u}||\underline{v}|\cos\theta$

Proof $\quad (\underline{v}-\underline{u}).\underline{u} = \underline{v}.\underline{u} - \underline{u}.\underline{u} = 0$

$$\underline{u}.\underline{v} = |\underline{u}|^2 = |\underline{u}||\underline{u}| = |\underline{u}||\underline{v}|.\frac{|\underline{u}|}{|\underline{v}|} = |\underline{u}||\underline{v}|\cos\theta \quad \square$$

Cosine rule



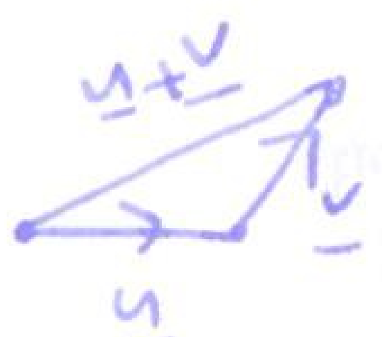
$$|\underline{u}-\underline{v}|^2 = |\underline{u}|^2 + |\underline{v}|^2 - 2|\underline{u}||\underline{v}|\cos\theta$$

Proof: $|\underline{u}-\underline{v}|^2 = (\underline{u}-\underline{v}).(\underline{u}-\underline{v}) = \underline{u}.\underline{u} - 2\underline{u}.\underline{v} + \underline{v}.\underline{v}$

$$= |\underline{u}|^2 + |\underline{v}|^2 - 2|\underline{u}||\underline{v}|\cos\theta. \quad \square.$$

special case: $\theta = \frac{\pi}{2} \quad |\underline{v}-\underline{u}|^2 = |\underline{u}|^2 + |\underline{v}|^2$ (Pythagoras).

§4.6 Triangle inequality



$$|\underline{u}+\underline{v}| \leq |\underline{u}| + |\underline{v}|.$$

usually derived from the Cauchy-Schwarz inequality $\quad |\underline{u}.\underline{v}| \leq |\underline{u}||\underline{v}|$

Proof (CS) $\quad \underline{u}.\underline{v} = |\underline{u}||\underline{v}|\cos\theta$

$|\underline{u}.\underline{v}| \leq |\underline{u}||\underline{v}|$ as $|\cos\theta| \leq 1$. $\square$.

Proof (triangle inequality)

$$|\underline{u}+\underline{v}|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})$$

$$= |\underline{u}|^2 + 2\underline{u}\cdot\underline{v} + |\underline{v}|^2$$

$$\leq |\underline{u}|^2 + 2|\underline{u}||\underline{v}| + |\underline{v}|^2 \quad (CS)$$

$$\leq (|\underline{u}| + |\underline{v}|)^2. \quad \square.$$

[why do it this way? CS holds in many spaces apart from $\mathbb{R}^2$].

Higher dimensional Euclidean spaces ($\mathbb{R}^n$)

$\mathbb{R}^n$: $(x_1, x_2, \dots, x_n)$

$\underline{u} = (u_1, \dots, u_n)$ $\quad \underline{v} = (\underline{v_1}, \dots, \underline{v_n})$

then $\underline{u}+\underline{v} = (u_1+v_1, \dots, u_n+v_n)$

$a\underline{u} = (au_1, \dots, au_n)$

$\underline{u}\cdot\underline{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$

define $|\underline{u}|^2 = \underline{u}\cdot\underline{u}$

$|\underline{u}| = \sqrt{u_1^2 + \dots u_n^2}$.

§4.7 Rotations matrices and complex numbers.

recall : $r_{c,s} : (x,y) \longmapsto (cx - sy, sx + cy)$ $\quad c = \cos\theta$, $s = \sin\theta$

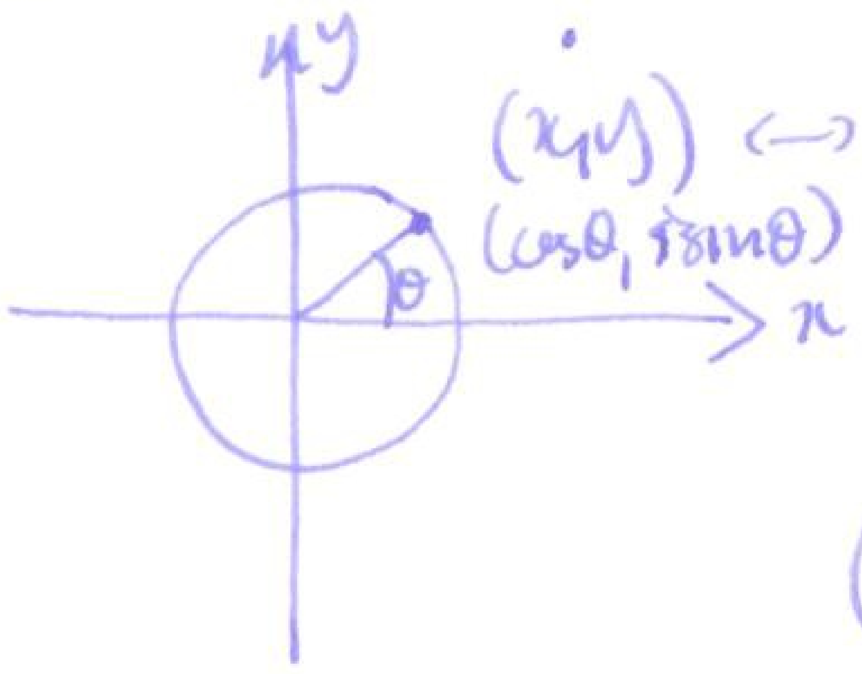
can also write : $\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

example rotation by θ_1 then rotation by θ_2

$$\begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix}\begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix} = \begin{bmatrix}\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2-\sin\theta_1\cos\theta_2 \\ \cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\end{bmatrix}$$

$$= \begin{bmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{bmatrix}.$$

Complex numbers



$(x,y) \longleftrightarrow \overset{z}{x+iy} \qquad i^2=-1$

$$(\cos\theta + i\sin\theta)\, z$$
$$= (\cos\theta + i\sin\theta)(x+iy)$$
$$= \cos\theta\, x + i\cos\theta\, y + i\sin\theta\, x - \sin\theta\, y$$
$$= \cancel{(\cos\theta-\sin\theta)x + i(\cos\theta+\sin\theta)y}$$
$$= cx - sy + i(sx+cy)$$

corresponds to rot^n by θ.

§7.2 Vector transformations

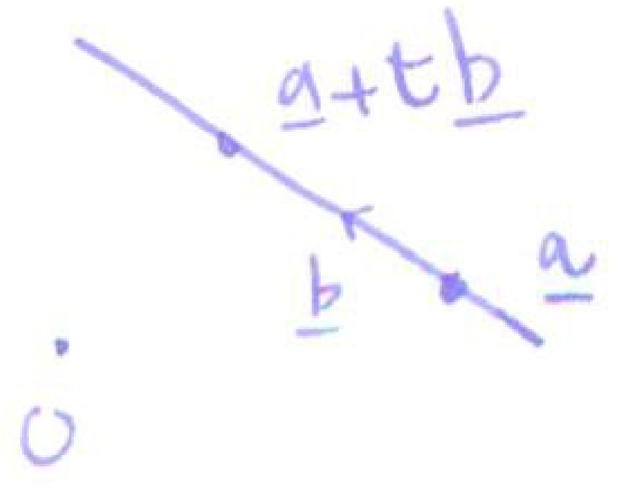
vector operations: addition $\underline{u}+\underline{v}$

scalar mult $a\underline{u}$

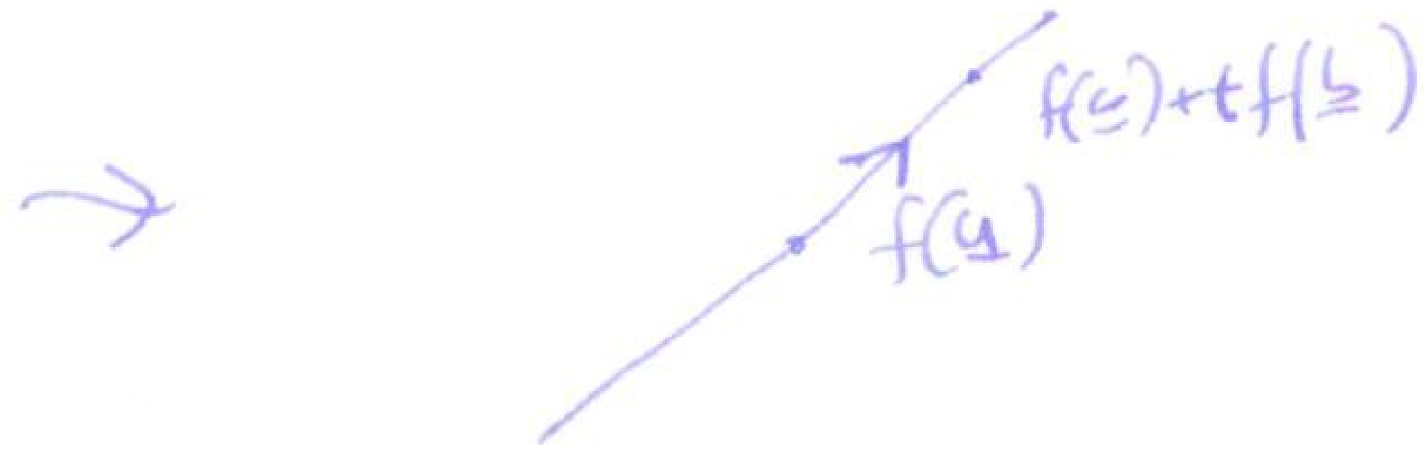
a linear transformation is a map between vector spaces which preserves addition and scalar multiplication i.e. $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$

$$f(\underline{u})+f(\underline{v}) = f(\underline{u}+\underline{v})$$
$$af(\underline{u}) = f(a\underline{u}).$$

claim linear transformations preserve lines



$$f(\underline{a}+t\underline{b}) = f(\underline{a}) + f(t\underline{b}) = f(\underline{a}) + t f(\underline{b})$$



still a straight line in direction $f(\underline{b})$.

so parallel lines $\mapsto$ parallel lines.

Matrix representation

every linear map $f: \mathbb{R}^2 \to \mathbb{R}^2$ is of the form

$$(x,y) \mapsto (ax+by, cx+dy).$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Proof $f((1,0)) = (a,c)$
$f((0,1)) = (b,d)$ say

$$\begin{aligned} f((x,y)) &= f(x(1,0) + y(0,1)) \\ &= x f(1,0) + y f((0,1)) \\ &= x(a,c) + y(b,d). \\ &= (ax, cx) + (by, dy) \\ &= (ax+by, cx+dy) \quad \square. \end{aligned}$$

Def^n determinant of $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(M) = ad - bc$.

Fact if $\det(M) \neq 0$ then $M^{-1} = \dfrac{1}{\det M} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Examples of linear transformations

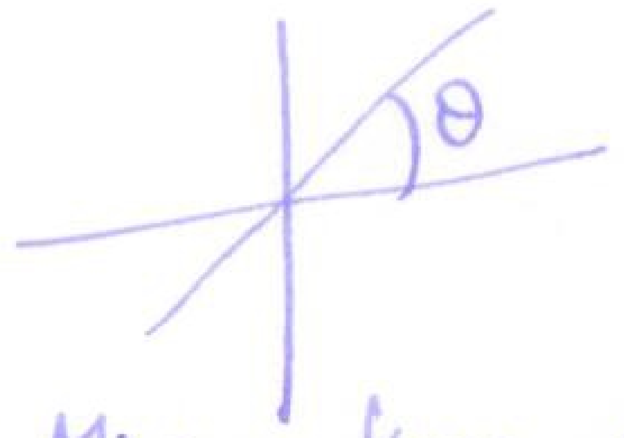
note $M(\underline{u}+\underline{v}) = M\underline{u} + M\underline{v}$ $\quad M a\underline{v} = a M\underline{v}$

rot^{ns} $(x,y) \mapsto (cx - sy, sx + cy)$ $\quad R_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

$\det R_\theta = c^2 + s^2 = 1$

so $R_\theta^{-1} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = R_{-\theta}$.

reflection in x-axis $(x,y) \mapsto (x,-y)$ $\quad \overline{X} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

reflection in  $R_\theta \overline{X} R_\theta^{-1}$

note: $\underline{v} \mapsto M\underline{v}$ fixes $\underline{0}$! $M\underline{0} = \underline{0}$.

not all linear transformations are isometries

example $S = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

stretch by factor k in x-direction.

Affine transformations

an affine transformation is a map of the form $g(\underline{u}) = M\underline{u} + \underline{c}$

(so we can do translations by affine maps!)

(but more general than $\mathrm{Isom}(\mathbb{R}^2)$)