so $\underline{w} - \underline{v} = \underbrace{c\underline{t} - b\underline{s} = a\underline{t} - a\underline{s}}$

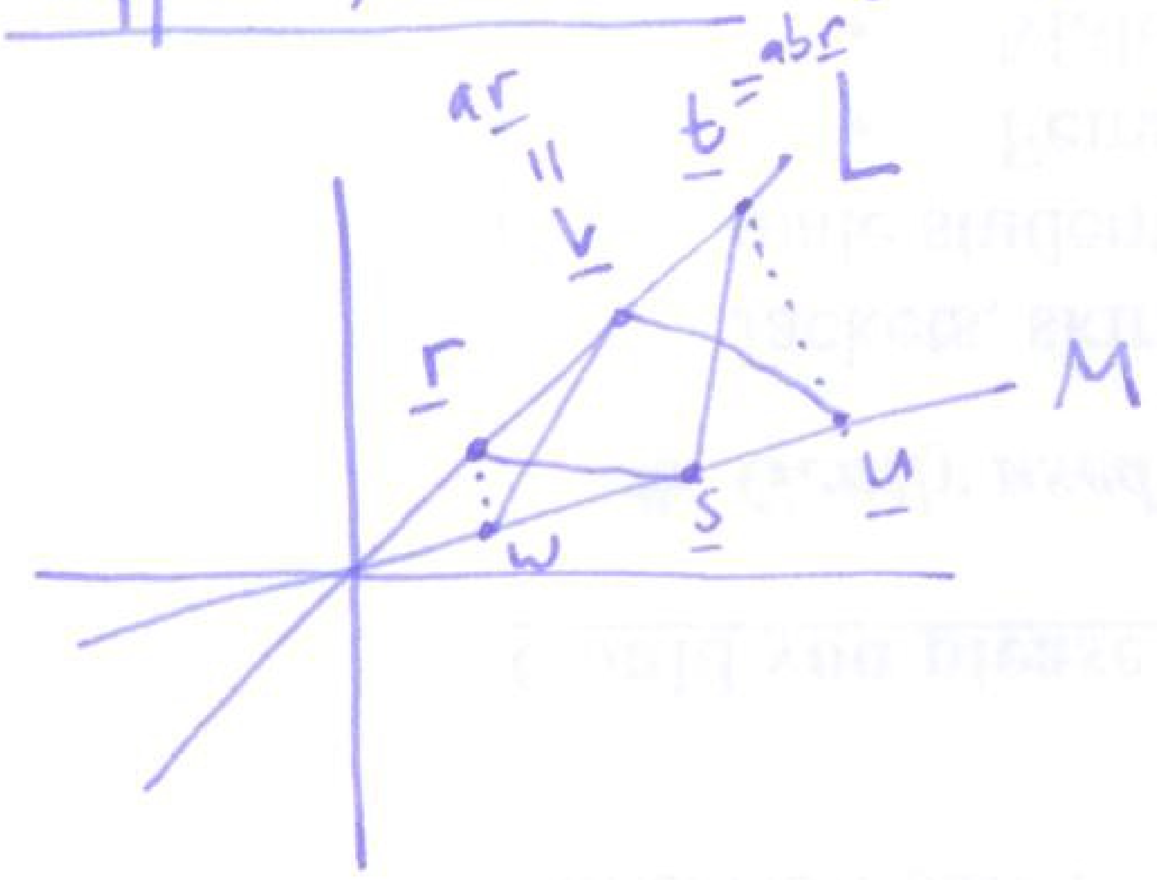
$$\underline{t}\,(c-a) = \underline{s}\,(a-b)$$

$$(c-a)\,\underline{t} + (b-a)\,\underline{s} = \underline{0}$$

but $\underline{s}, \underline{t}$ point in different directions, so $\left.\begin{array}{l} c-a=0 \\ b-a=0 \end{array}\right\}$ so $\begin{array}{l} c=a \\ b=a \end{array}$

so $\underline{v} = a\underline{s}$ and $\underline{w} = a\underline{t}$ as required $\square$.

Pappus theorem (vector version)



if $\underline{u} - \underline{v}$ parallel to $\underline{s} - \underline{r}$

and $\underline{t} - \underline{s}$ parallel to $\underline{v} - \underline{w}$

then $\underline{u} - \underline{t}$ parallel to $\underline{w} - \underline{r}$.

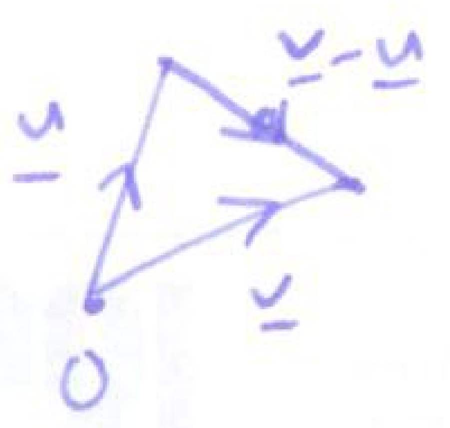
Proof. on L: $\underline{r}$, $\underline{v} = a\underline{r}$, $\underline{t} = b\underline{v} = ab\,\underline{r}$

Thales: $\underline{u} = a\underline{s}$, $\underline{s} = b\underline{w}$

$\underline{u} = ab\,\underline{w}$

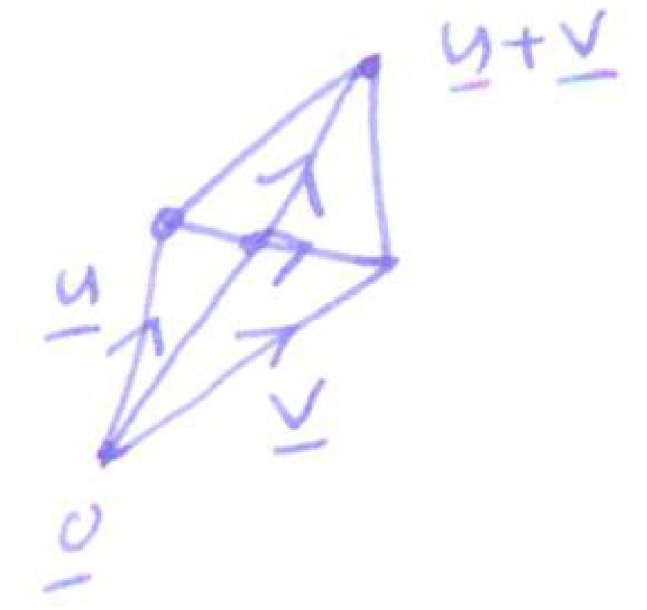
so $\underline{u} - \underline{t} = ab\,\underline{w} - ab\,\underline{r} = ab\,(\underline{w} - \underline{r})$ $\square$.

§4.3 Midpoints and centroids



midpoint is $\underline{u} + \frac{1}{2}(\underline{v} - \underline{u}) = \frac{1}{2}(\underline{u} + \underline{v})$. (vector average)

Thm Diagonals of a parallelogram bisect each other.

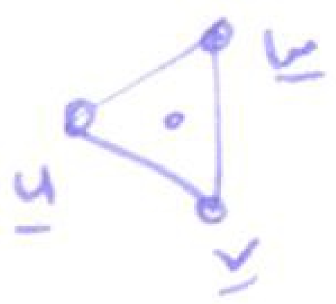


midpoint of of $\underline{u}+\underline{v}$ is $\frac{1}{2}(\underline{u}+\underline{v})$

midpoint of ~~u~~ $\underline{v}-\underline{u}$ is $\frac{1}{2}(\underline{u}+\underline{v})$

same point! □

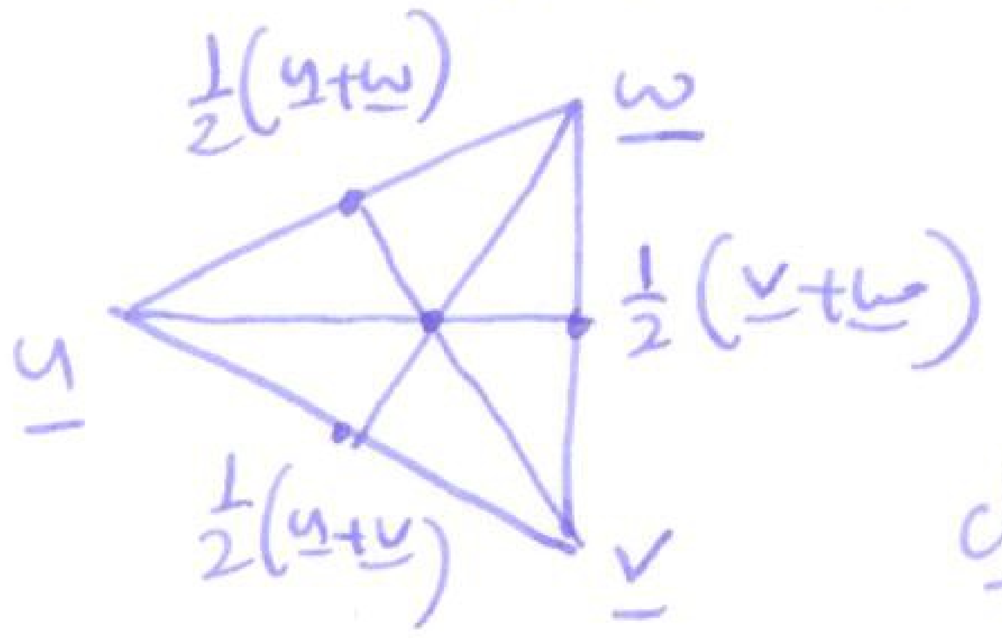
vector average also known as <u>barycenter</u>, <u>center of mass</u>, <u>centroid</u>.



$\frac{1}{3}(\underline{u}+\underline{v}+\underline{w})$

Thm The medians of any triangle pass through the same point, the centroid of the triangle.

<u>Proof</u>



<u>claim</u> the common point lies $\frac{2}{3}$ of the way along the median.

<u>check</u>: $\frac{2}{3}\frac{1}{2}(\underline{u}+\underline{w})+\frac{1}{3}\underline{v} = \frac{1}{3}(\underline{u}+\underline{v}+\underline{w})$.

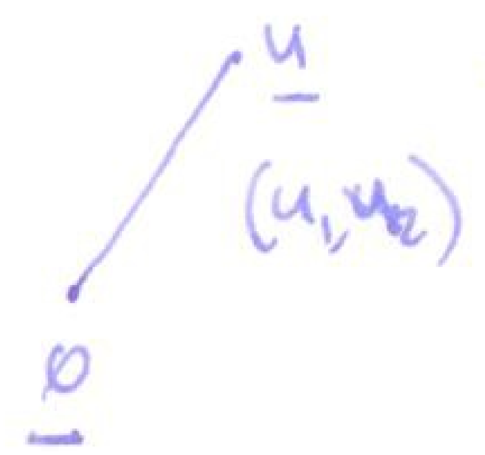
similarly $\frac{2}{3}\frac{1}{2}(\underline{w}+\underline{v})+\frac{1}{3}\underline{u} = \frac{1}{3}(\underline{u}+\underline{v}+\underline{w})$.

$\frac{2}{3}\frac{1}{2}(\underline{u}+\underline{v})+\frac{1}{3}\underline{w} = \frac{1}{3}(\underline{u}+\underline{v}+\underline{w})$. □.

§4.4 Inner product

$\underline{u} = (u_1, u_2)$ $\underline{v} = (v_1, v_2)$ then the <u>inner product</u> or <u>scalar product</u> of $\underline{u}$ and $\underline{v}$ is $\underline{u}.\underline{v} = u_1v_1 + u_2v_2$ (number not vector!)

useful properties:
- $\underline{u}.\underline{v} = \underline{v}.\underline{u}$
- $\underline{u}.(\underline{v}+\underline{w}) = \underline{u}.\underline{v} + \underline{u}.\underline{w}$
- $(a\underline{u}).\underline{v} = u.(a\underline{v}) = a(\underline{u}.\underline{v})$



• length $|\underline{u}| = \sqrt{u_1^2 + u_2^2} = \sqrt{\underline{u}.\underline{u}}$

so $|\underline{u}|^2 = \underline{u}.\underline{u}$

$$|\underline{v}-\underline{u}|^2 = (\underline{v}-\underline{u}).(\underline{v}-\underline{u}) = \underline{v}.\underline{v} - 2\underline{u}.\underline{v} + \underline{u}.\underline{u}$$
$$= |\underline{v}|^2 + |\underline{u}|^2 - 2\underline{u}.\underline{v}$$

• $\underline{u}, \underline{v}$ perpendicular iff $\underline{u}.\underline{v} = 0$

<u>recall</u> perpendicular iff product of slopes is -1 (§3.5)

slope $\underline{u}$ is u_2/u_1 so $\dfrac{u_2}{u_1}\dfrac{v_2}{v_1} = -1$

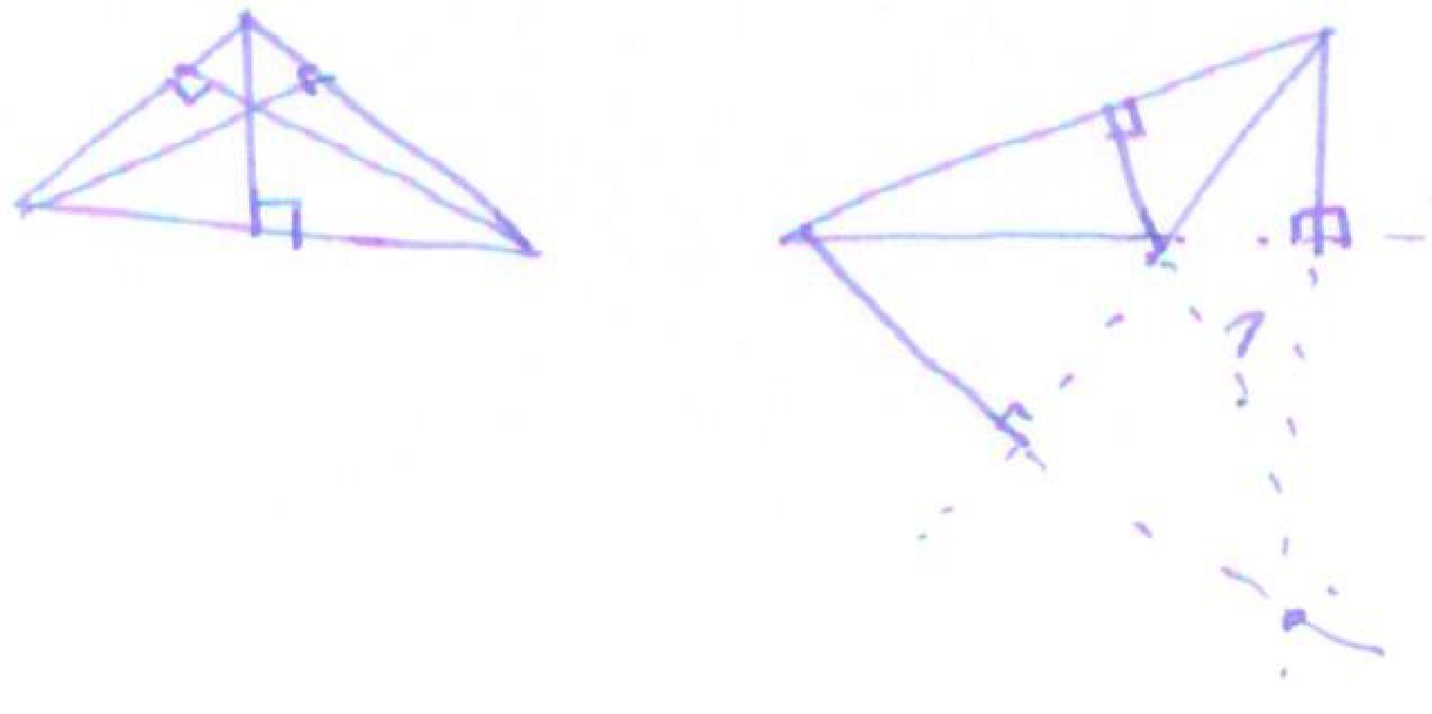
$\underline{v}$ $\quad v_2/v_1$

$u_2 v_2 = -u_1 v_1$

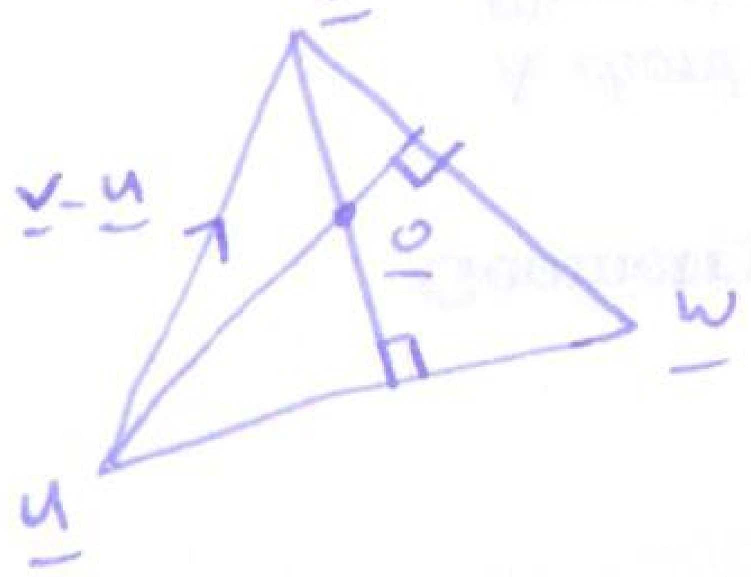
$u_1 v_1 + u_2 v_2 = 0 = \underline{u}.\underline{v}$.

<u>Thm</u> <u>Concurrence of altitudes</u>

In any triangle, the perpendiculars (altitudes) from vertices to opposite sides, have a common point.



<u>Proof</u> Take $\underline{0}$ to be the intersection of two of the altitudes:



<u>want</u> $\underline{w} \perp$ to $\underline{v}-\underline{u}$

$\underline{u} \perp$ to $\underline{w}-\underline{v}$, i.e. $\underline{u}.(\underline{w}-\underline{v}) = 0$ ①

$\underline{v} \perp$ to $\underline{w}-\underline{u}$ i.e. $\underline{v}.(\underline{w}-\underline{u}) = 0$ ②

① − ② : $\underline{u}.\underline{w} - \underline{u}.\underline{v} - \underline{v}.\underline{w} + \underline{u}.\underline{v} = 0$