Let S be a set (e.g. $\mathbb{R}^2$)

a <u>transformation</u> of S is a function $S \to S$

a collection G of transformations forms a <u>group</u> if

- if $f, g \in G$ then $f \circ g \in G$
- if $f \in G$ then f has an inverse f^{-1} which is also in G.

<u>observation</u> $\Rightarrow$ $id_S \in G$ as $ff^{-1} = id$.

Klein's view of geometry: study the transformation groups and their invariants.

fix x-axis: rot^n by π about <u>any</u> point on x-axis

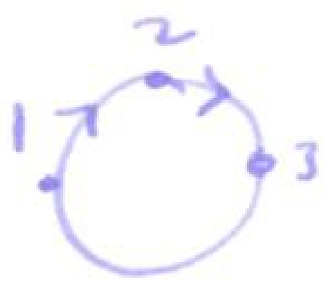
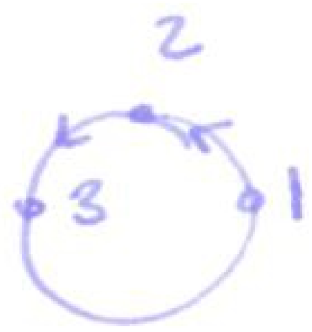
space of all lines $S^1 \times \mathbb{R}$ ~~...~~ or $S^1 \times \mathbb{R}$.

<u>Example</u> $\mathbb{R}^2$, G = Isometries of $\mathbb{R}^2$

invariants: distance, straight lines $\mapsto$ straight lines
circles $\mapsto$ circles.

perpendicular lines $\mapsto$ perpendicular lines
angles are preserved.

non-invariants: vertical lines.

clockwise order on the circle  reflect 

<u>Oriented Euclidean geometry</u> $\mathrm{Isom}^+(\mathbb{R}^2) \subset \mathrm{Isom}(\mathbb{R}^2)$

all isometries which are a product of an <u>even</u> number of reflections.

(i.e. rotations + translations)

<u>claim</u> $\mathrm{Isom}^+(\mathbb{R}^2)$ is a group.

check:
- f, g products of an even number of reflections, then so is fg.
- if $f = r_1 \cdots r_{2n}$ product of an even number of reflections then $f^{-1} = r_{2n} \cdots r_1$ also product of an even number of reflections.

claim $Isom^+(\mathbb{R}^2)$ preserves clockwise order.

proof: a single reflection reverses clockwise order, so two preserve it, so an even number of reflections preserve it. $\square$.

§4.1 Vectors

Examples $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n \ldots$

rules for vectors:

$$\underline{u}+\underline{v} = \underline{v}+\underline{u}$$
$$\underline{u}+(\underline{v}+\underline{w}) = (\underline{u}+\underline{v})+\underline{w}$$
$$\underline{u}+\underline{0} = \underline{u}$$
$$\underline{u}+(-\underline{u}) = \underline{0}$$

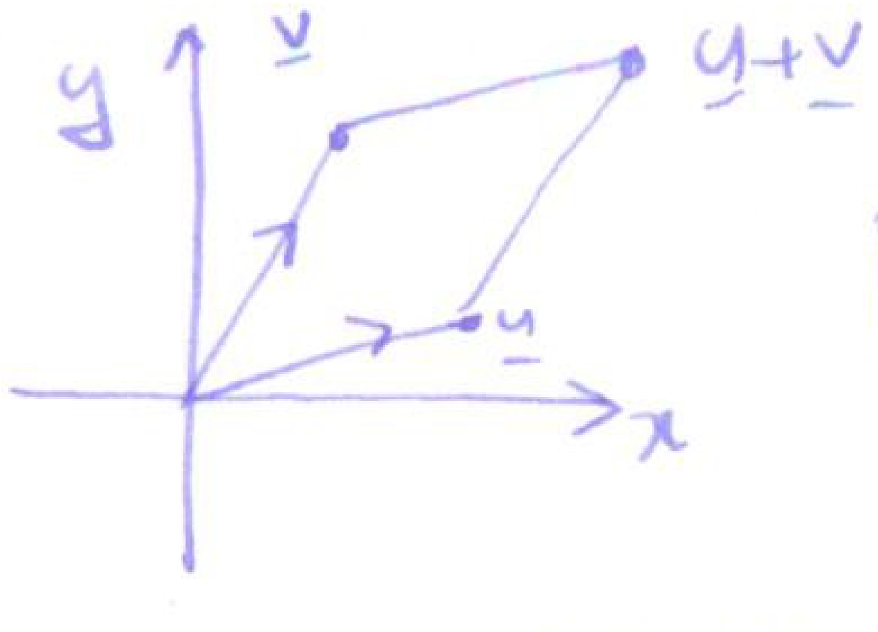
$$1\underline{u} = \underline{u}$$
$$a(\underline{u}+\underline{v}) = a\underline{u} + a\underline{v}$$
$$(a+b)\underline{u} = a\underline{u} + b\underline{u}$$
$$a(b\underline{u}) = (ab)\underline{u} .$$

rules hold for $\underline{u},\underline{v} \in \mathbb{R}$.

$\underline{u},\underline{v} \in \mathbb{R}^2$ if we define $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$

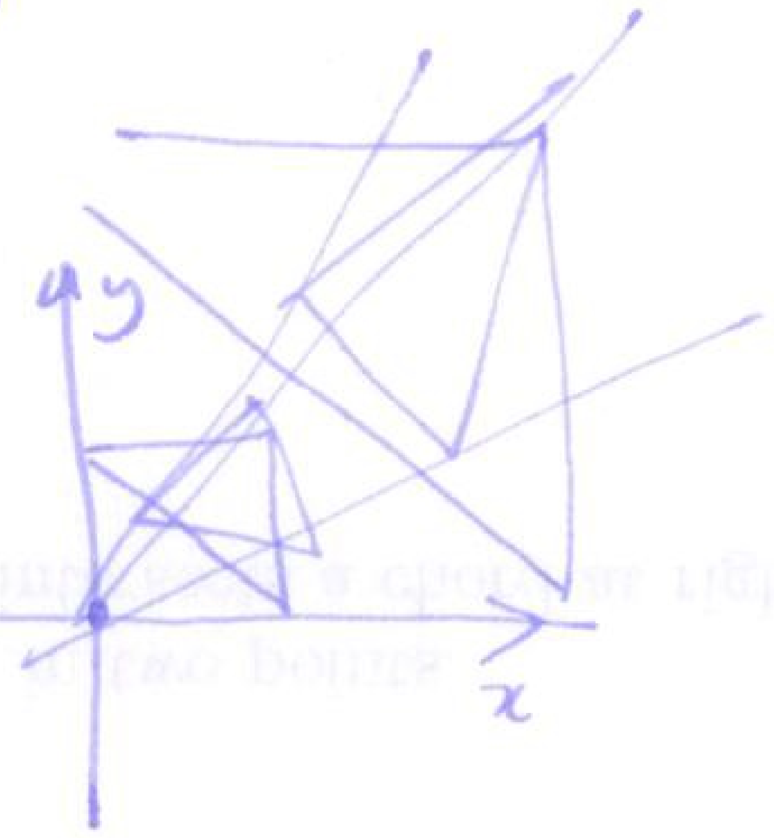
and $a(x_1,y_1) = (ax_1, ay_1)$

geometrically: sum.



parallelogram rule for vector sum.

scalar multiplication: scaling by a



$(x,y) \mapsto (2x, 2y)$.

Def$^{\underline{n}}$ A (real) vector space V is a set with operations of addition and scalar multiplication s.t.

- if $\underline{u},\underline{v} \in V$ then $\underline{u}+\underline{v} \in V$ and $a\underline{u} \in V$
- there is a zero vector $\underline{0}$ s.t. $\underline{u}+\underline{0}=\underline{u}$ for all $\underline{u} \in V$

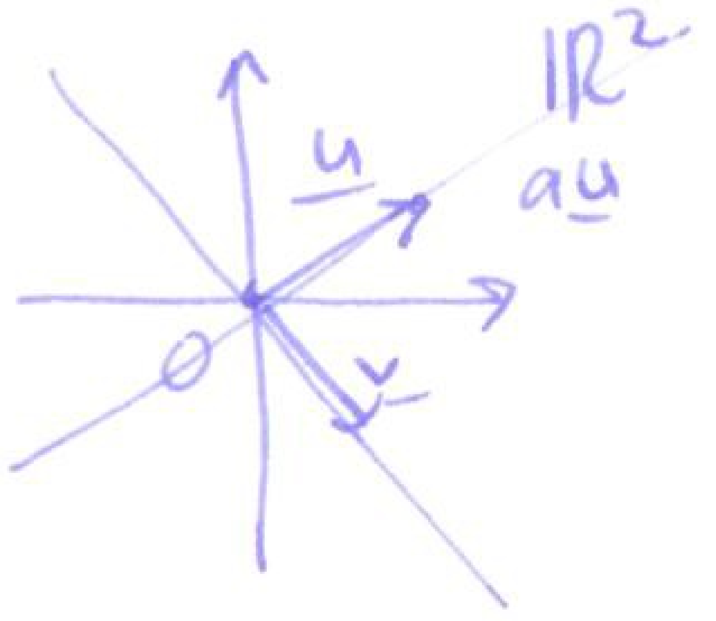
- addition and scalar multiplication have the eight properties listed above.

<u>examples</u> $\mathbb{R}^3$: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$a(x_1, y_1, z_1) = (ax_1, ay_1, az_1)$.

$\mathbb{R}^n$ similarly...

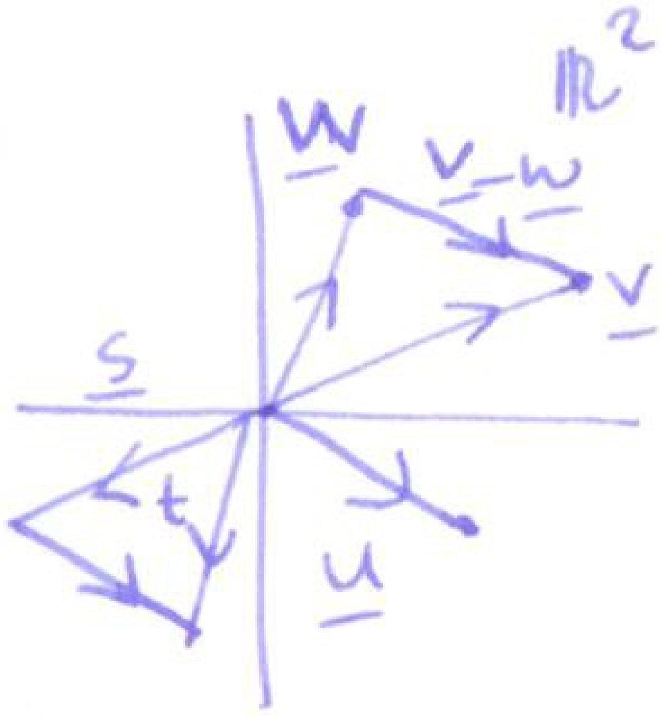
§4.2 Direction and linear independence



given a non-zero vector $\underline{u}$, the vectors $a\underline{u}$, $a \in \mathbb{R}$ gives the line through $\underline{0}$ and $\underline{u}$.

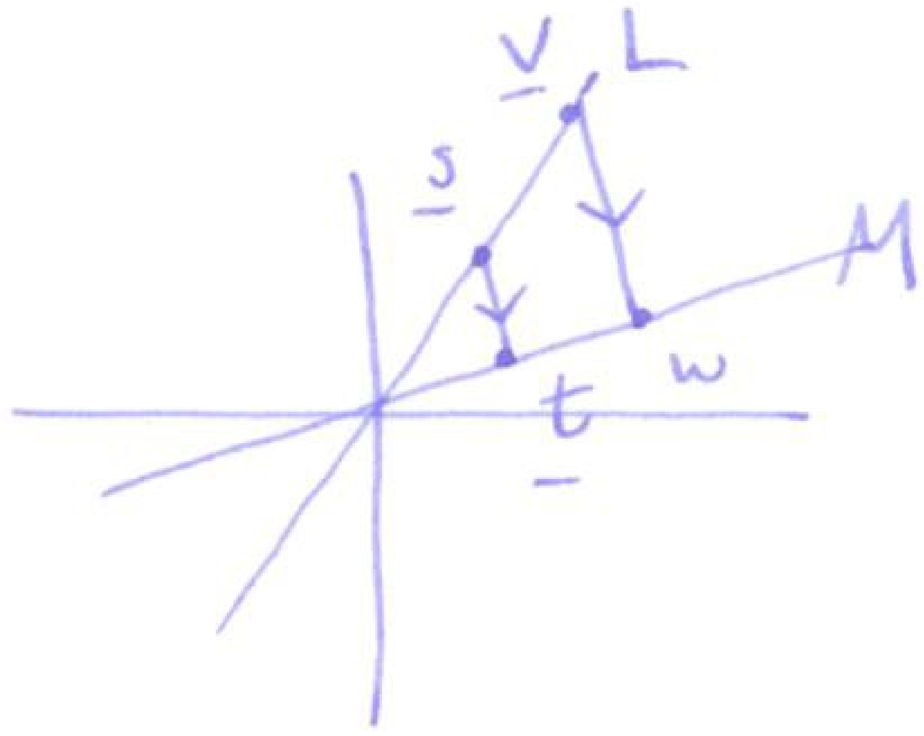
two vectors $\underline{u}, \underline{v}$ point in different directions if neither is a multiple of the other.

we say $\underline{u}, \underline{v}$ are <u>linearly independent</u> if there are no real numbers a, b s.t. $a\underline{u} + b\underline{v} = \underline{0}$.



$\underline{v}-\underline{w}$ is the relative direction from $\underline{w}$ to $\underline{v}$.

two relative directions are parallel if $\underline{v}-\underline{w} = a(\underline{s}-\underline{t})$.



<u>Thales theorem</u> (vector version)

if $\underline{s}, \underline{v}$ lie on a line L through $\underline{0}$ and $\underline{t}, \underline{w}$ lie on a different line M through $\underline{0}$, and $\underline{w}-\underline{v}$ is parallel to $\underline{t}-\underline{s}$, then $\underline{v} = a\underline{s}$ and $\underline{w} = a\underline{t}$ for some number a.

<u>Proof</u> parallel : $\underline{w}-\underline{v} = a(\underline{t}-\underline{s})$

$\underline{s}, \underline{v}$ on same line : $\underline{v} = b\underline{s}$

$\underline{t}, \underline{w}$ on same line : $\underline{w} = c\underline{t}$