

## §3.9 Discussion

(24)  
(25)

coordinates enable:

- description of curves by equations, use algebra to study geometry  
(algebraic geometry)
- linear algebra
- transformations, from isometry groups, characterize geometry

## §7.1 The isometry group of the plane

$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is an isometry if it preserves distances, i.e.

$$d(f(p), f(\alpha)) = d(p, \alpha).$$

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

- composition of isometries is an isometry:

then  $f \circ g$  is an isometry

$$\text{check: } d(f(g(p)), f(g(\alpha))) = d(g(p), g(\alpha)) = d(p, \alpha).$$

- inverse of an isometry is an isometry

Proof: recall every isometry is the product of at most three reflections.

$$f = r_1 r_2 r_3 \quad r_i: \text{reflections.}$$

a reflection has an inverse (itself!)  $r_i r_i = \text{id}$

claim:  $f^{-1} = r_3 r_2 r_1$  is an inverse for  $f$ .

check

$$\text{Proof: } f \circ r_3 r_2 r_1 = r_1 r_2 r_3 \circ r_3 r_2 r_1 = r_1 r_2 r_3 r_3 r_2 r_1$$

$$= r_1 r_2 r_1 = r_1 r_1 = \text{id. } \square$$

$f^{-1}$  is an isometry as it is a product of reflections.  $\square$

warning: order matters:  $r_1 r_2 \neq r_2 r_1$  (find example!)

Let  $S$  be a set (e.g.  $\mathbb{R}^2$ )

a transformation of  $S$  is a function  $S \rightarrow S$

a collection  $G$  of transformations forms a group if

- if  $f, g \in G$  then  $f \circ g \in G$
- if  $f \in G$  then  $f$  has an inverse  $f^{-1}$  which is also in  $G$ .

observation  $\Rightarrow \text{id}_S \in G$  as  $ff^{-1} = \text{id}$ .

Klein's view of geometry: study the transformation groups and their invariants.

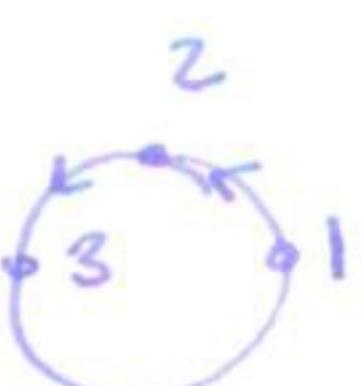
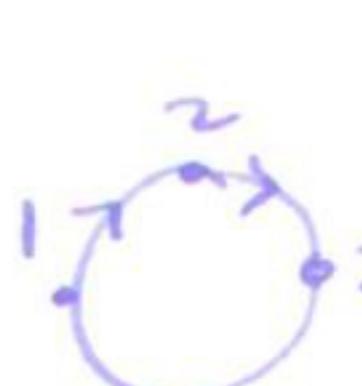
example  $\mathbb{R}^2$ ,  $G = \text{Isometries of } \mathbb{R}^2$

invariants: distance, straight lines  $\mapsto$  straight lines  
circles  $\mapsto$  circles.

perpendicular lines  $\mapsto$  perpendicular lines  
angles are preserved.

non-invariants: vertical lines.

clockwise order on the circle



reflect

Oriented Euclidean geometry  $\text{Isom}^+(\mathbb{R}^2) \subset \text{Isom}(\mathbb{R}^2)$

all isometries which are a product of an even number of reflections.  
(i.e. rotations + translations)

claim  $\text{Isom}^+(\mathbb{R}^2)$  is a group.

check:

- $f, g$  products of an even number of rotations, then so is  $f \circ g$ .

• if  $f = r_1 \dots r_m$  product of an even number of reflections  
then  $f^{-1} = r_m \dots r_1$  also product of an even number of reflections.