§3.8 Discussion

coordinates enable:

- description of curves by equations, use algebra to study geometry (algebraic geometry)
- linear algebra
- transformations, ~~tran~~ isometry groups, characterize geometry

§7.1 The isometry group of the plane

$f: \mathbb{R}^2 \to \mathbb{R}^2$ is an <u>isometry</u> if it preserves distances, i.e.

$$d(P, Q) = d(f(P), f(Q)). \qquad d(\overset{P}{(x_1, y_1)}, \overset{Q}{(x_2, y_2)}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

- <u>composition of isometries is an isometry</u>: $f: \mathbb{R}^2 \to \mathbb{R}^2$ $g: \mathbb{R}^2 \to \mathbb{R}^2$ isometries, then $f \circ g$ is an isometry

<u>check</u>: $d(f(g(P)), f(g(Q))) = d(g(P), g(Q)) = d(P, Q)$.

- <u>inverse of an isometry is an isometry</u>

<u>Proof</u>: recall every isometry is the product of at most three reflections.

$f = r_1 r_2 r_3 \qquad r_i$ reflections.

a reflection has an inverse (itself!) $\quad r_i r_i = id$

<u>claim</u>: $f^{-1} = r_3 r_2 r_1$ is an inverse for f.

<u>check</u> ~~Proof~~: $f \circ r_3 r_2 r_1 = r_1 r_2 r_3 \circ r_3 r_2 r_1 = r_1 r_2 \overbrace{\underbrace{r_3 r_3}}^{id} r_2 r_1$

$$= r_1 \underbrace{r_2 r_2}_{id} r_1 = r_1 r_1 = id. \quad \square$$

f^{-1} is an isometry as it is a product of reflections.

<u>warning</u>: order matters: $r_1 r_2 \neq r_2 r_1$ (find example!)

let S be a set (e.g. $\mathbb{R}^2$)

a <u>transformation</u> of S is a function $S \to S$

a collection G of transformations forms a <u>group</u> if

- if $f, g \in G$ then $f \circ g \in G$
- if $f \in G$ then f has an inverse f^{-1} which is also in G.

<u>observation</u> $\Rightarrow$ $id_S \in G$ as $ff^{-1} = id$.

Klein's view of geometry: study the transformation groups and their invariants.

<u>Example</u> $\mathbb{R}^2$, $G =$ Isometries of $\mathbb{R}^2$

invariants: distance, straight lines $\mapsto$ straight lines
circles $\mapsto$ circles.

perpendicular lines $\mapsto$ perpendicular lines
angles are preserved.

non-invariants: vertical lines.
clockwise order on the circle

<u>Oriented Euclidean geometry</u> $\mathrm{Isom}^+(\mathbb{R}^2) \subset \mathrm{Isom}(\mathbb{R}^2)$

all isometries which are a product of an <u>even</u> number of reflections.

(i.e. rotations + translations)

<u>claim</u> $\mathrm{Isom}^+(\mathbb{R}^2)$ is a group.

check:
- f, g products of an even number of reflections, then so is fg.
- if $f = r_1 \dots r_{2m}$ product of an even number of reflections then $f^{-1} = r_{2m} \dots r_1$ also product of an even number of reflections.