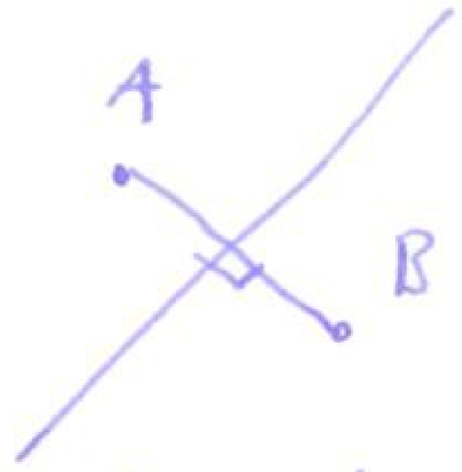
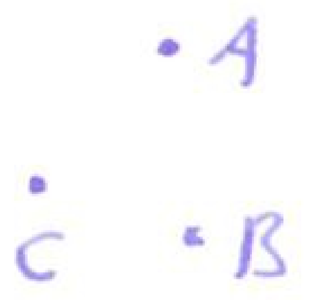
§3.7 Three ~~&~~ reflections theorem

recall: equidistant set between two points is a straight line:



[say: how much infer, eg $(x,y) \mapsto (x+a, y+b)$].

Thm An isometry of $\mathbb{R}^2$ is determined by the images of 3 points which don't lie on a common line.

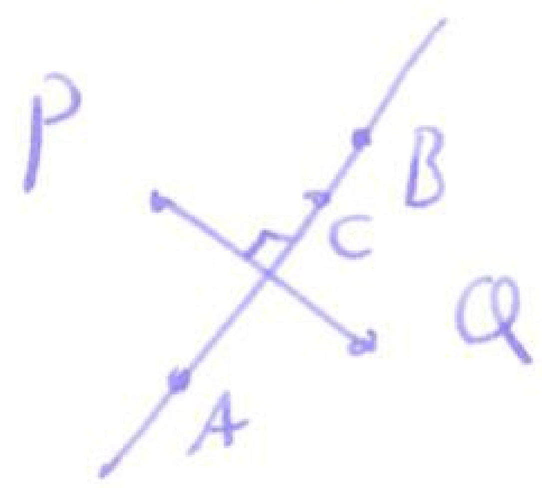


Proof let A, B, C be 3 points which don't lie on a common line

claim: any point $P \in \mathbb{R}^2$ is determined by its distances from A, B and C

proof (of claim): suppose two points P, Q have same distances to A, B, C

i.e. $|AP| = |AQ|$, $|BP| = |BQ|$ and $|CP| = |CQ|$.



then A, B, C all lie in the equidistant set for P and Q

$\Rightarrow$ A, B, C lie in a line $\#$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry, so preserves distances.

so distance from ~~P~~ P to A same as distance of $f(P)$ to $f(A)$.

P	B	$f(P)$ $f(B)$
P	C	$f(P)$ $f(C)$

so we know distance of $f(P)$ to each of $f(A), f(B)$ and $f(C)$, so this information determines $f(P)$ uniquely $\square$.

[say: how do we find $f(P)$ if we know distances].

Thm (Three reflections theorem)

Any isometry of $\mathbb{R}^2$ is a combination of at most three reflections.

Proof choose 3 points A, B, C not on a line

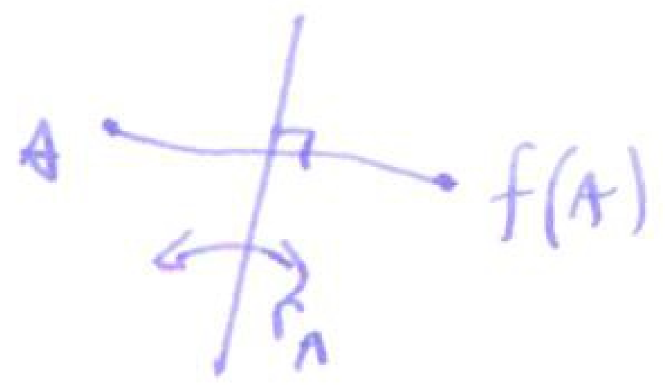
an isometry f is determined by $f(A), f(B), f(C)$

aim: we will construct a sequence of (at most 3) reflections which takes $A \mapsto f(A)$

$B \mapsto f(B)$

$C \mapsto f(C)$

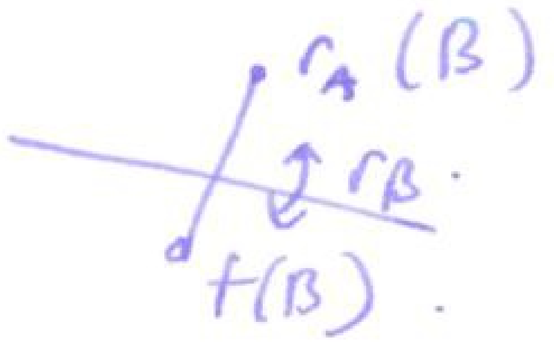
Let r_A be reflection in the equidistant line between A and $f(A)$

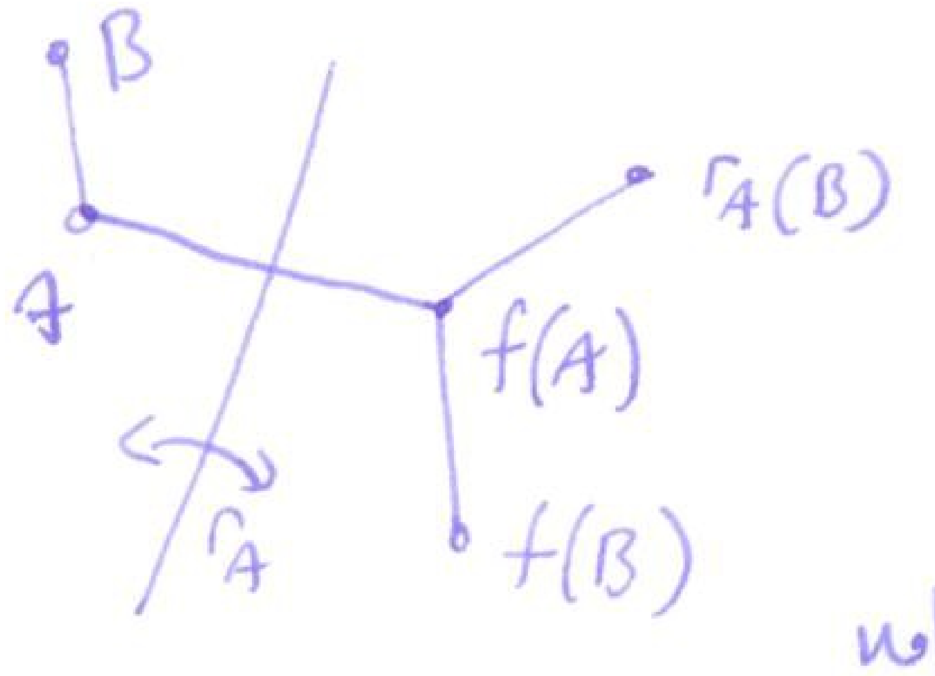


$r_A : A \mapsto f(A)$
$r_A : B \mapsto r_A(B)$
$r_A : C \mapsto r_A(C)$

if $r_A(B) = f(B)$ then $r_A : B \to f(B)$ as well.

if $r_A(B) \neq f(B)$ then let r_B be reflection in the equidistant line between $r_A(B)$ and $f(B)$, and consider $r_B r_A$





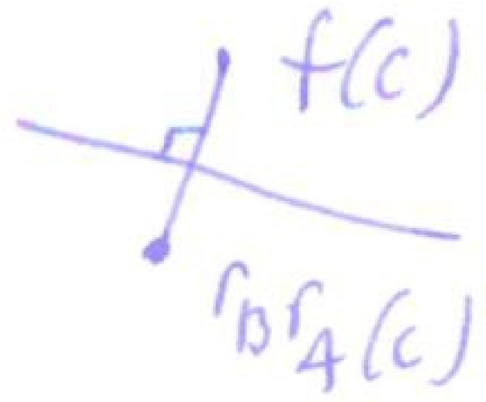
$A \mapsto r_B(f(A))$
$r_B r_A : B \longmapsto r_B r_A(B) = f(B)$
$C \mapsto r_B r_A(C)$

<u>note</u> : $|f(A)\, r_A(B)| = |f(A)\, f(B)| = |AB|$.
$|f(A)\, f(B)| = |AB|$.

so $f(A)$ lies on equidistant line between $r_A(B)$ and $f(B)$, so r_B fixes $f(A)$, i.e. $r_B(f(A)) = f(A)$. so $r_B r_A$: $A \mapsto f(A)$
$B \mapsto f(B)$
$C \mapsto r_B r_A(C)$.

if $r_B r_A(C) = f(C)$ done : $f = r_B r_A$.

if $r_B r_A(C) \neq f(C)$ choose r_C to be reflection in equidistant line between $r_B r_A(C)$ and $f(C)$



~~takes~~ so $r_C r_B r_C(C) = f(C)$

$r_C r_B r_A$: $A \mapsto r_C(f(A))$
$B \mapsto r_C(f(B))$
$C \mapsto f(C)$

<u>note</u> : $|f(A)\, f(C)| = |AC|$
~~$|f(A)\, f(C)| = |AC|$~~.
$|f(A)\, r_B r_A(C)| = |\cancel{f(A}\; r_B r_A(A)\, r_B r_A(C)| = |AC|$

so A lies on equidistant line through $f(C), r_B r_A(C)$.

similarly $|f(B)\, f(C)| = |BC|$
$|f(B)\, r_B r_A(C)| = |r_B r_A(B)\, r_B r_A(C)| = |BC|$

so B lies on equidistant line through $f(C), r_B r_A(C)$.

so $r_C r_B r_A : \; A \mapsto f(A)$
$B \mapsto f(B)$
$C \mapsto f(C)$

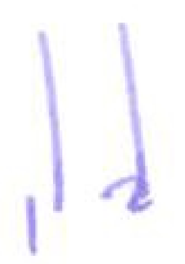
and if two isometries take A, B, C to $f(A), f(B), f(C)$ they are the same,

so $f = r_C r_B r_A$. □

<u>Corollary</u> every isometry is a ~~reflection~~, translation, rotation, reflection or glide reflection.

<u>Proof</u> r_1 : reflection

$r_2 r_1$:

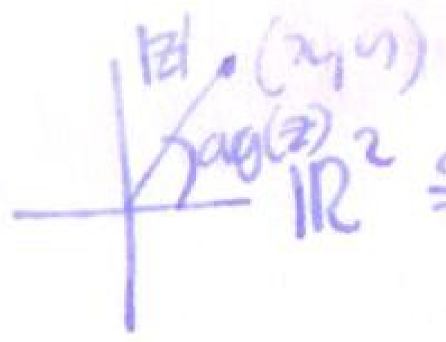


translation

rotation.

$r_3 r_2 r_1$:

<u>complex numbers</u>.



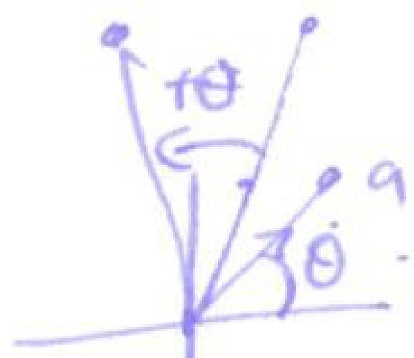
$(x,y) \leftrightarrow x+iy \qquad i^2=-1.$

$\mathbb{R}^2 \cong \mathbb{C}$

useful facts : $(a+bi)+(c+di) = (a+c)+i(b+d)$
$(a+bi)(c+di) = (ac-bd)+i(bc+ad)$

check $\overline{zw} = \bar{z}\bar{w}$ $\rightarrow$ $z = x+iy$ then $\bar{z} = x-iy$. so $z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$

$\frac{1}{z} = \frac{1}{x+iy} = \frac{\bar{z}}{z\bar{z}} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$.



addition $\leftrightarrow$ translations. $z \mapsto z+a$

multiplication : $z \mapsto az$: scale by a factor of $|a|$ and add $\arg(a)$ to $\arg(z)$.

<u>rotation</u> choose a with $|a|=1$.

$z = x+iy = \cos\theta + i\sin\theta$.

<u>Fact</u> $e^{i\theta} = \cos\theta + i\sin\theta$.

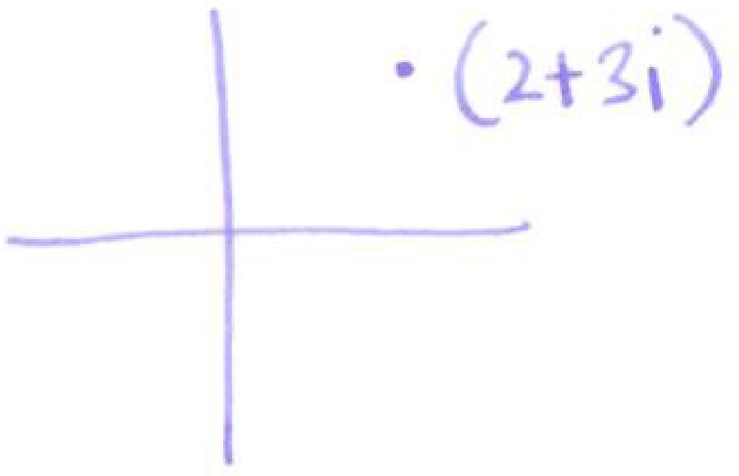
check $\overline{e^{i\theta}} = e^{-i\theta}$

$\left[e^{x+iy} = e^{x}(\cos y + i\sin y)\right]$..

so rotations given by $z \mapsto e^{i\theta} z$
(about 0)

reflection in x-axis $z \mapsto \bar{z}$

$x+iy \mapsto x-iy$



rotation about $(2+3i) = a$

$$\underset{\text{translate}}{z \longmapsto z - (\overset{a}{\cancel{2+3i}})} \underset{\text{rotate.}}{\longmapsto} e^{i\theta}(z-a) \underset{\text{translate}}{\longmapsto} e^{i\theta}(z-a)+a$$

$$= e^{i\theta}z + a - ae^{i\theta}.$$

in fact any $z \mapsto e^{i\theta}z + a$ is a rotation about

fixed point: $z = e^{i\theta}z + a$ $\quad z = \dfrac{a}{1-e^{i\theta}}$.

general reflection / glide reflection $z \mapsto \bar{z} + r$

$f: z \mapsto e^{i\theta}z + a$ has inverse $f^{-1}: z \mapsto e^{-i\theta}(z-a) = e^{-i\theta}z - ae^{-i\theta}$.

$$z \longmapsto e^{-i\theta}(z-a) \xmapsto{\text{reflect}} \overline{e^{-i\theta}(z-a)} = e^{i\theta}(\bar{z}-\bar{a}) \mapsto e^{i\theta}(e^{i\theta}(\bar{z}-\bar{a}))+a$$

$$z \mapsto e^{2i\theta}(\bar{z}-\bar{a})+a = e^{2i\theta}\bar{z} + a - \bar{a}e^{2i\theta}.$$

$$z \mapsto e^{i\theta}z+a \mapsto \overline{e^{i\theta}z+a} + r = e^{-i\theta}\bar{z} + \bar{a} + r \mapsto (e^{-i\theta}\bar{z} + \bar{a} + r - a)e^{-i\theta}$$

$$= e^{-2i\theta}\bar{z} + e^{-i\theta}(\underbrace{\bar{a}-a}_{\text{imaginary}} + r)$$

so $e^{i\theta}z + b$ is a glide reflection iff $e^{i\theta/2}b$ is purely imaginary