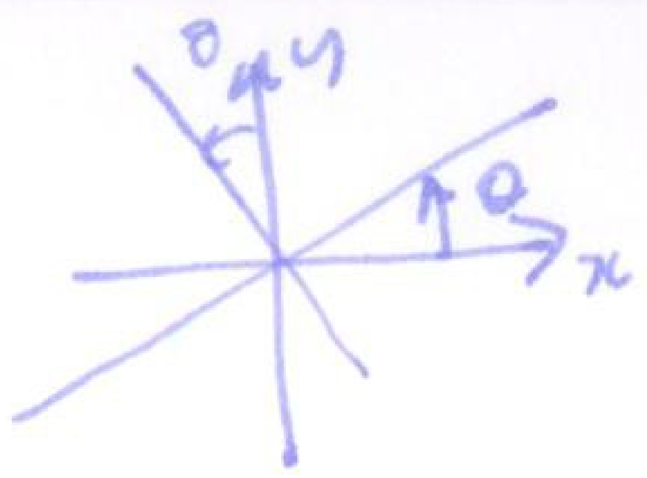
Rotations



convenient to work with $(\cos\theta, \sin\theta) = (c,s)$
so $c^2+s^2=1$.

claim rotation is given by $r_{c,s} : (x,y) \longmapsto (cx-sy,\ sx+cy)$

check this is an isometry: $P(x_1,y_1)$ $Q(x_2,y_2)$ $r_{c,s}(P)\,(cx_1-sy_1,\ sx_1+cy_1)$
$r_{c,s}(Q)\,(cx_2-sy_2,\ sx_2+cy_2)$

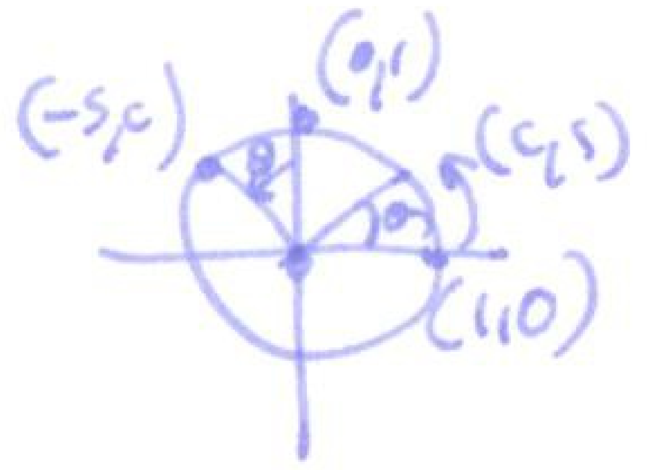
$$|r_{c,s}(P)\, r_{c,s}(Q)| = \sqrt{(cx_2-sy_2-cx_1+sy_1)^2+(sx_2+cy_2-sx_1-cy_1)^2}$$

$$= \sqrt{(c(x_2-x_1)-s(y_2-y_1))^2+(s(x_2-x_1)^2+c(y_2-y_1)^2)^2}$$

$$= \sqrt{\begin{aligned}&c^2(x_2-x_1)^2-2cs(x_2-x_1)(y_2-y_1)+s^2(y_2-y_1)^2\\ &+s^2(x_2-x_1)^2+2cs(x_2-x_1)(y_2-y_1)+c^2(y_2-y_1)^2\end{aligned}}.$$

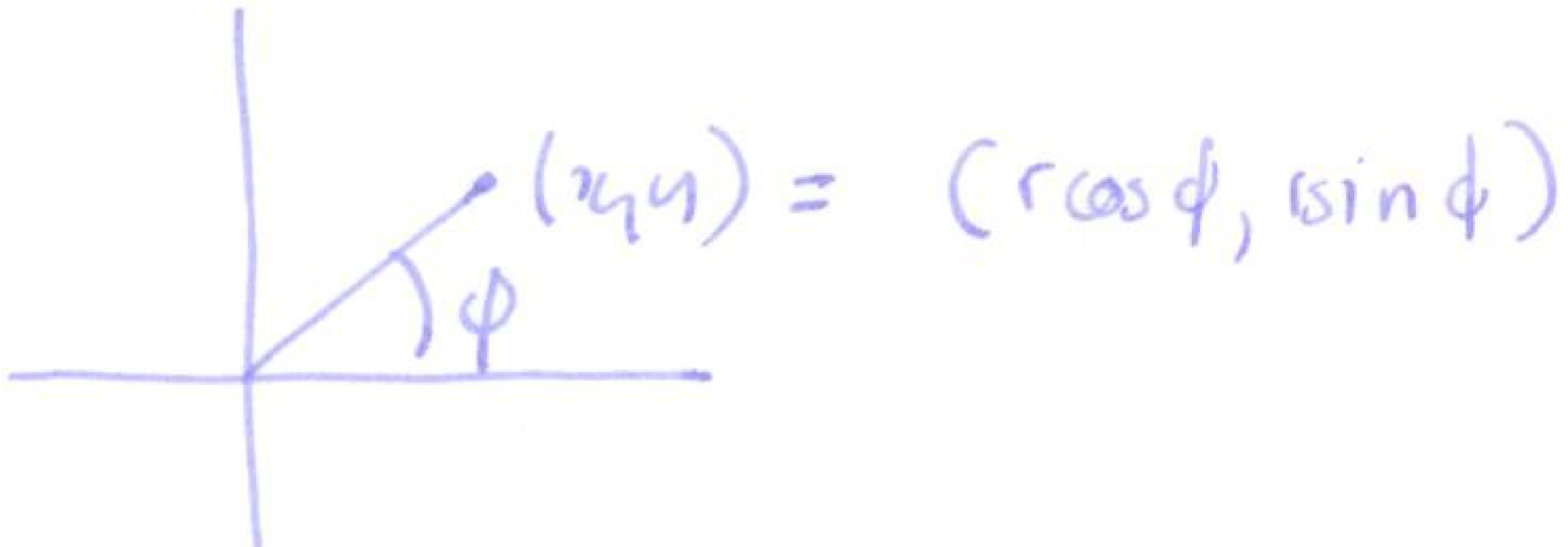
$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = |PQ|.$$

note: $r_{c,s}$ fixes $O=(0,0)$ sends $(1,0)\mapsto(c,s)=(\cos\theta,\sin\theta)$
$(0,1)\mapsto(-s,c)=(-\sin\theta,\cos\theta)$



fact: isometry is determined by where it sends 3 points.

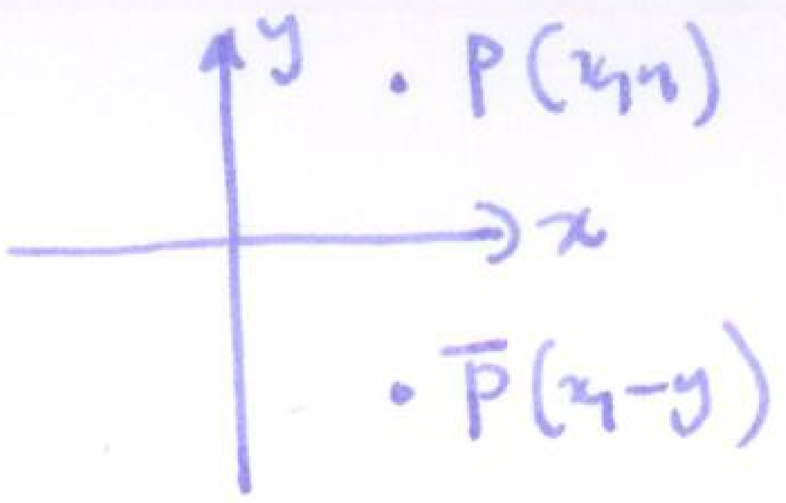
manual check:



$$r_{c,s}(x,y) = (cx-sy,\ sx+cy) = (\cos\theta\, r\cos\phi - \sin\theta\, r\sin\phi,\ \sin\theta\, r\cos\phi + \cos\theta\, r\sin\phi)$$
$$= (r(\cos\theta\cos\phi-\sin\theta\sin\phi),\ r(\sin\theta\cos\phi+\cos\theta\sin\phi))$$
$$= (r\cos(\theta+\phi),\ r\sin(\theta+\phi)).$$

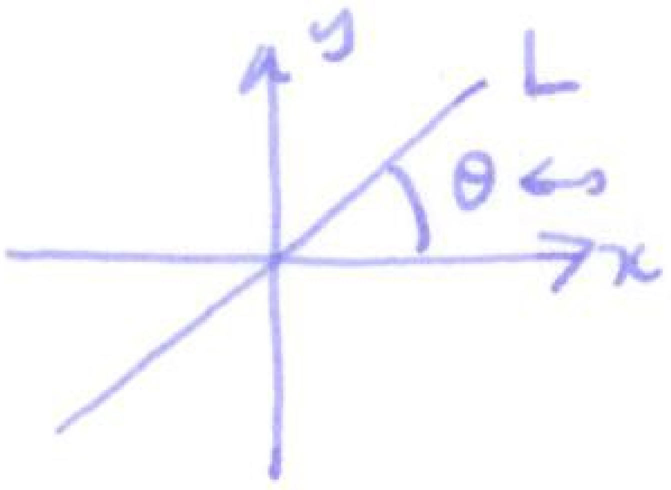
observation: we can now take any line to any other line by translation/rotation

Reflections



check: $(x,y) \mapsto (x,-y)$ is an isometry.

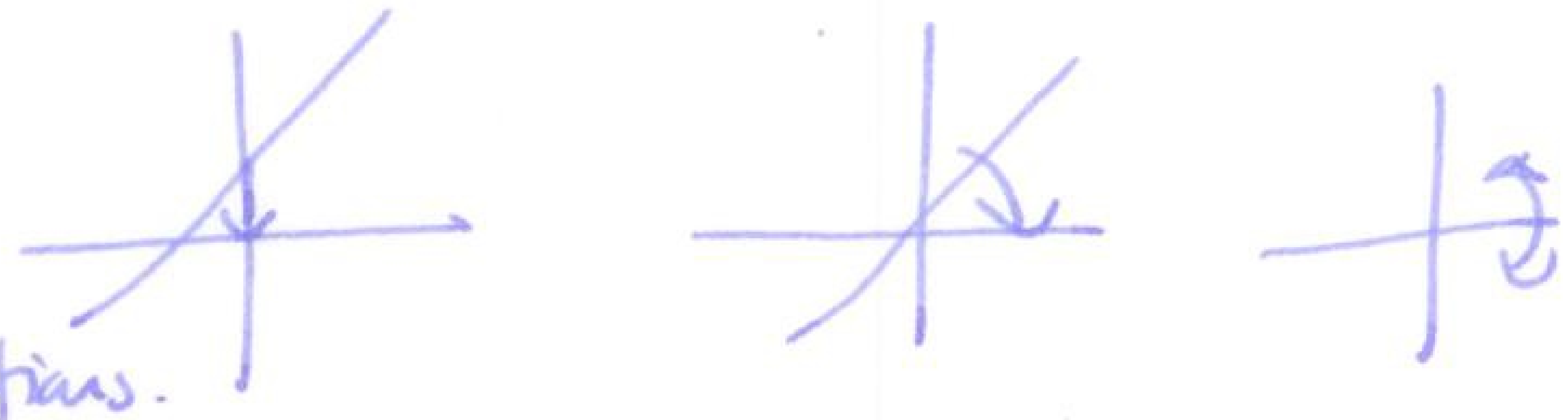
reflect in line through origin:



$(c,s) = (\cos\theta, \sin\theta)$

- rotate L to x-axis $\quad r_{(c,-s)}$
- do reflection in x-axis $\quad \rho: (x,y) \mapsto (x,-y)$
- rotate x-axis back to L $\quad r_{(c,s)}$

in general: translate, then rotate.

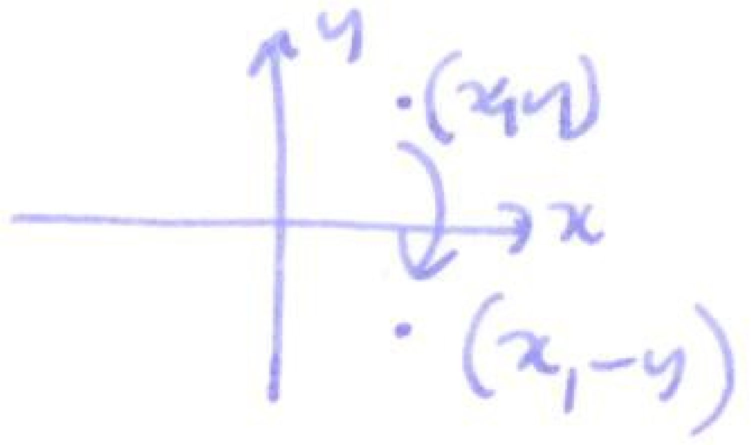


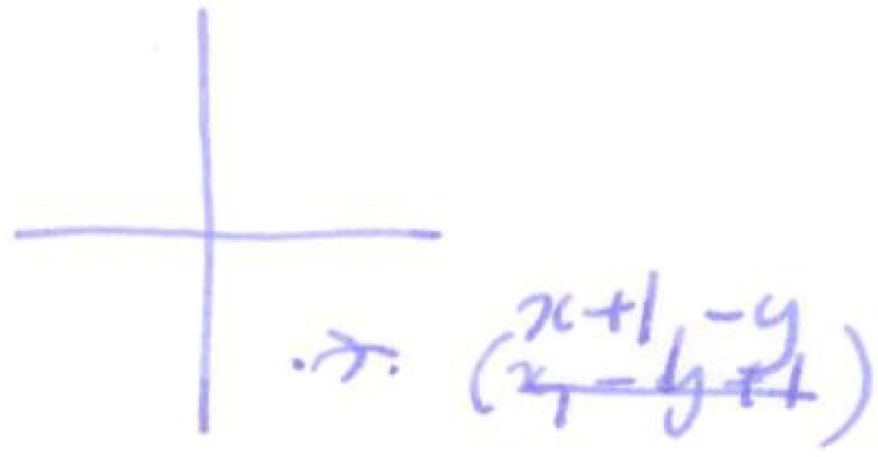
fact: every isometry product of (at most 3) reflections.

Glide reflections

a glide reflection is a reflection followed by a translation in the direction of the line of reflection.

example:





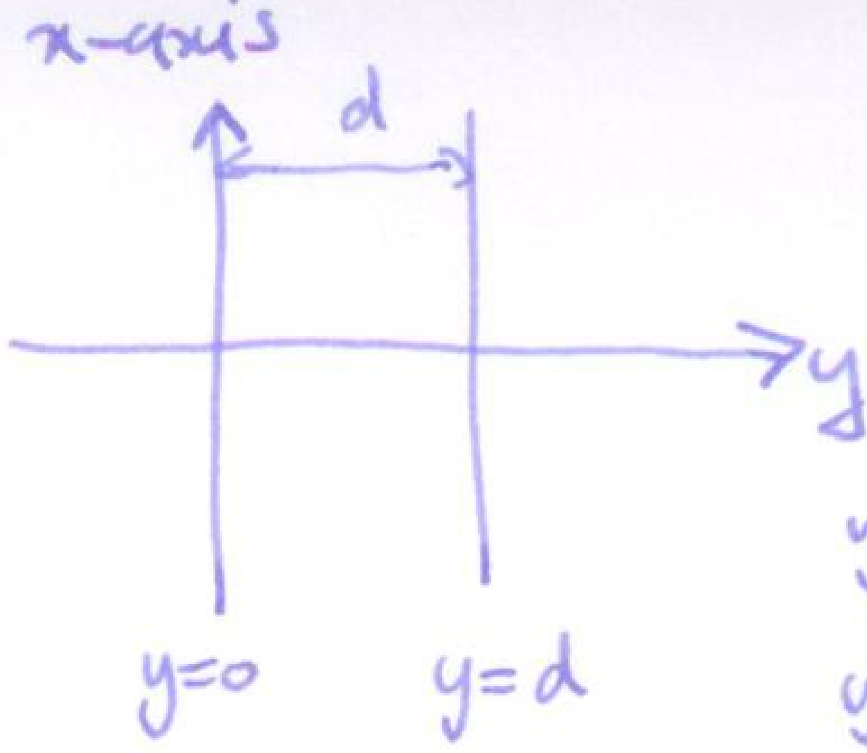
example
$(x,y) \mapsto (x+1, -y)$

note:
- not a translation, as translation maps any line in direction of translation to itself
- not a rotation, as a rotation has a fixed point (glide reflection does not
- not a reflection, as a reflection has fixed points.

notation $\quad (x,y) \mapsto (ax+by+c, dx+ey+f) \quad$ or $\quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \underset{\uparrow \, 2\times 2 \text{ matrix.}}{A} \begin{bmatrix} x \\ y \end{bmatrix} + \underline{c}$

or $\quad z = x+iy \quad z \mapsto az+b$
or $a\bar{z}+b$.

Examples



① reflections in two parallel lines. gives a translation. of distance

$y=0$: $(x,y) \mapsto (-x, y)$

$y=d$: $(x,y) \mapsto$?

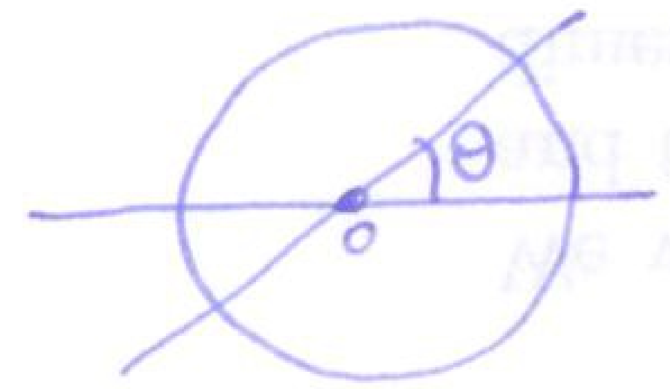
$y = d$ ~~translation~~ reflection : translate : $(x,y) \mapsto (x-d, y)$

then reflect $((x,y) \mapsto (-x,y)) \qquad \mapsto (-x+d, y)$.

then translate back $(x,y) \mapsto (x+d, y) \qquad \mapsto (-x+2d, y)$.

so $(x,y) \xmapsto{y=0} (-x,y) \xmapsto{y=d} (x+2d, y)$

② reflection in two intersecting lines gives a rotation. (of 2θ, θ angle at which lines meet).



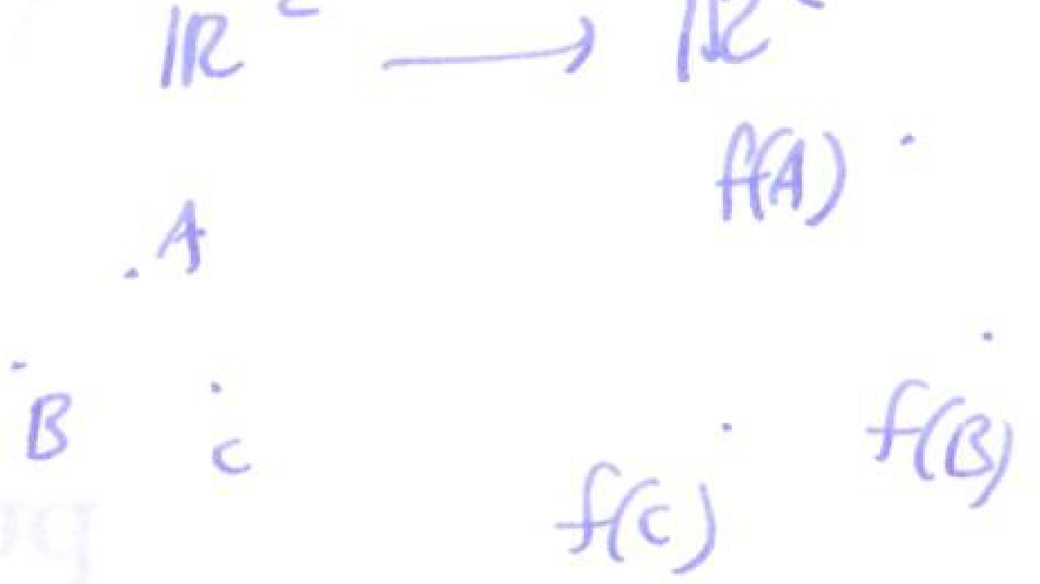
(can check this by calculation)

(easiest in $\mathbb{C}$-notation)

§3.7. Three reflections theorem

recall : equidistant set between two points is a straight line

Thm An isometry of $\mathbb{R}^2$ is determined by the images of 3 points which don't lie on a common line.



Proof Let A, B, C be points which don't lie on a common line.

claim any point $P \in \mathbb{R}^2$ is determined by its distances from A, B and C.

Proof spse $Q \in \mathbb{R}^2$ has same distance to A, B and C.
Then A, B, C all lie in the equidistant set of P, Q which is a line #.