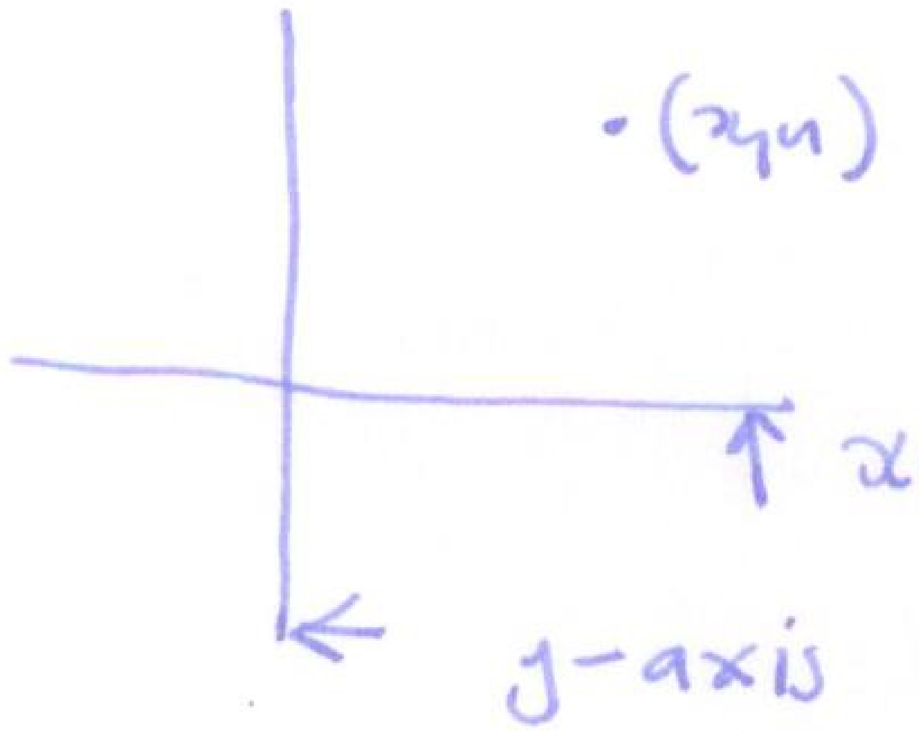
picture :

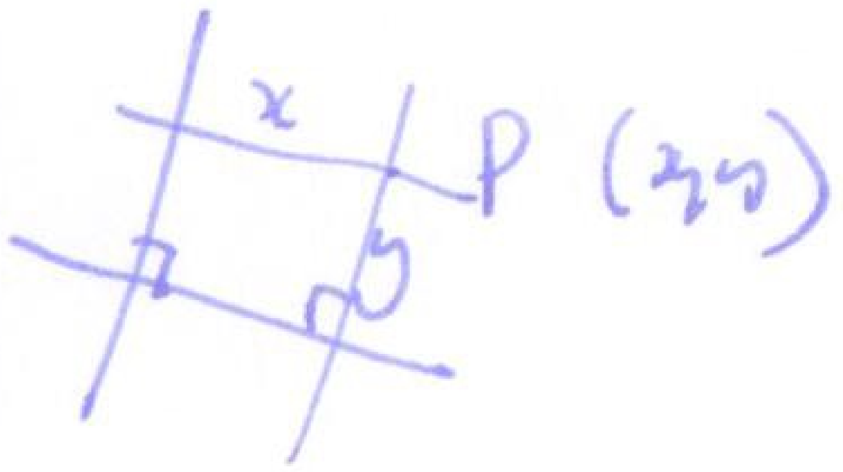


↑ x-axis (pairs $(x,0)$) or $y=0$

← y-axis (pairs $(0,y)$, or $x=0$).

two approaches

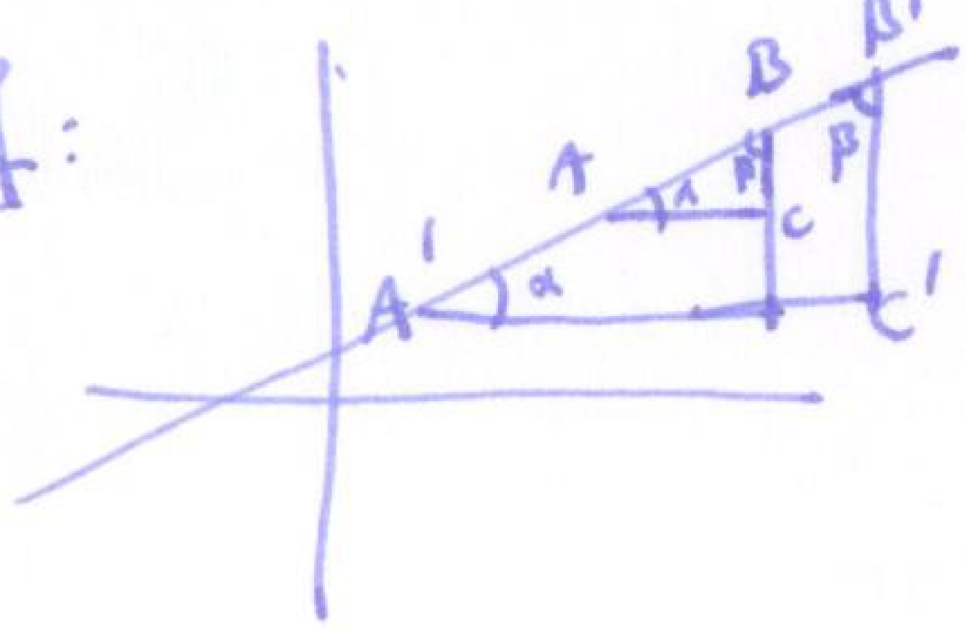
① coordinatize Euclidean geometry: given a plane, draw a pair of perpendicular lines. Now for any point P, draw perpendiculars to each line and measure distances, this gives P coordinates (x,y)



now we can investigate properties of geometric constructs in the coordinate system.

Example straight lines have constant slope.

Proof:

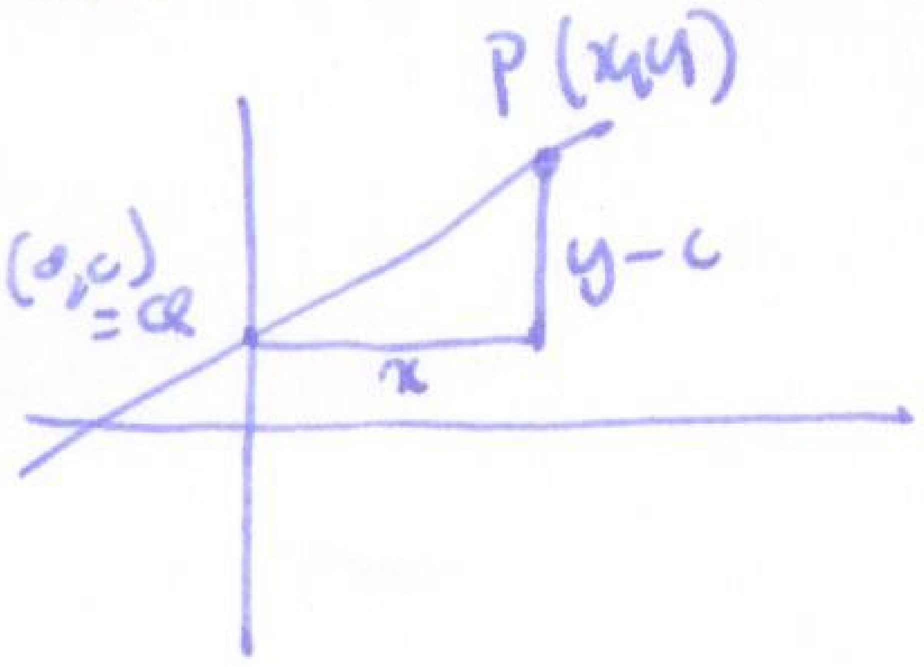


$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

triangles ABC and $A'B'C'$ are similar (same angles) so corresponding sides are proportional, i.e.

$$\frac{BC}{AC} = \frac{B'C'}{A'C'} = \text{slope}.$$

equation for a line



$$\text{slope} = a = \frac{y-c}{x}$$

$$\left.\begin{array}{ll} & y = ax+c \\ \text{straight lines} & x = c \end{array}\right\}$$ all lines have one of these forms.

symmetric form : $ax+by+c=0$ } all lines have an equation of this form.

(called a linear equation)

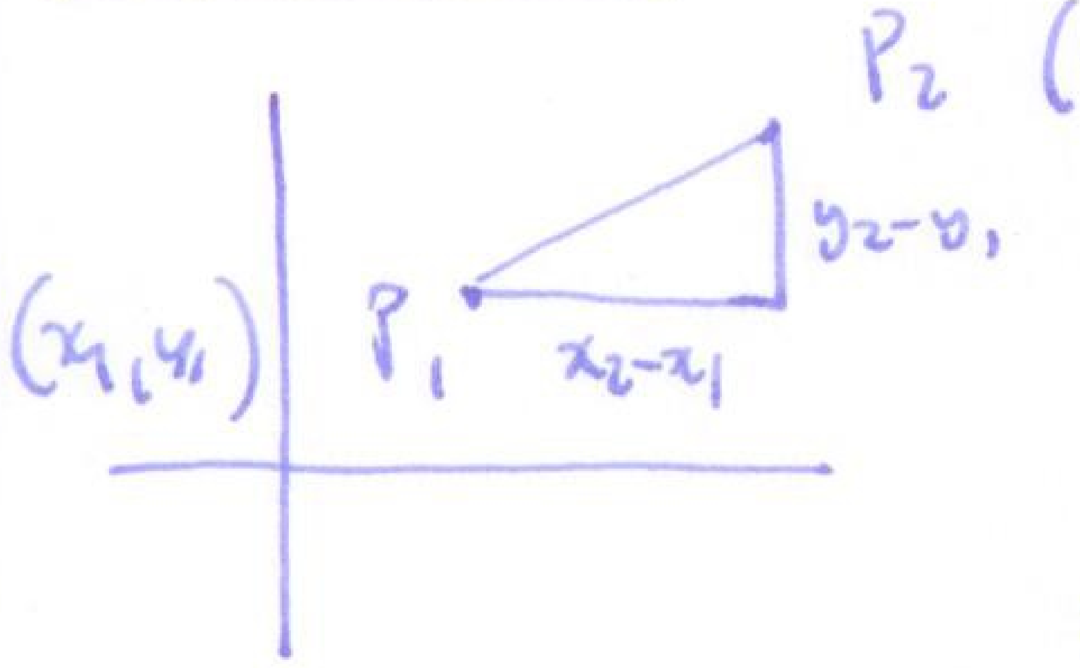
but now we can do the following

(2) define lines as solutions to linear equations in $\mathbb{R}^2$, and check they satisfy Euclid's axioms. (exercises!).

so $\mathbb{R}^2$, linear equations in (x,y) give a model for Euclidean geometry

§3.3 Distance

from euclidean geometry:



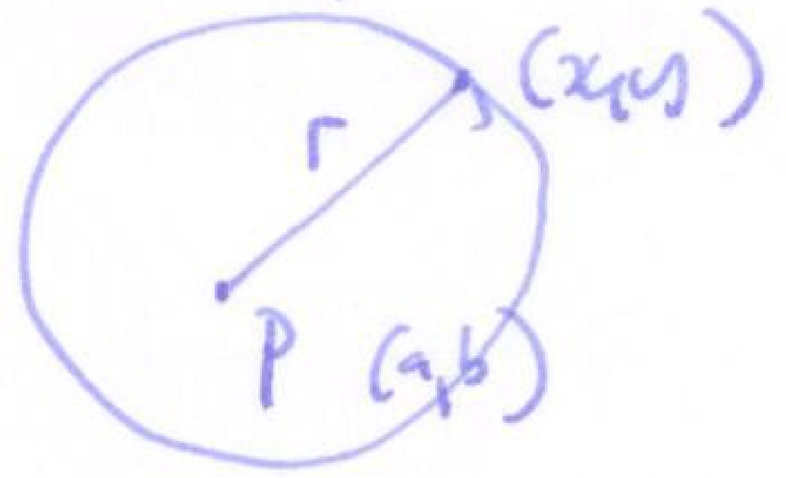
Pythagoras: $|P_1P_2|^2 = (x_2-x_1)^2 + (y_2-y_1)^2$

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)} .$$

so we can define the distance between two points (x_1,y_1) and (x_2,y_2) to be $\sqrt{(x_2-x_1)^2-(y_2-y_1)^2}$

and then check it has required properties of Euclidean distance.

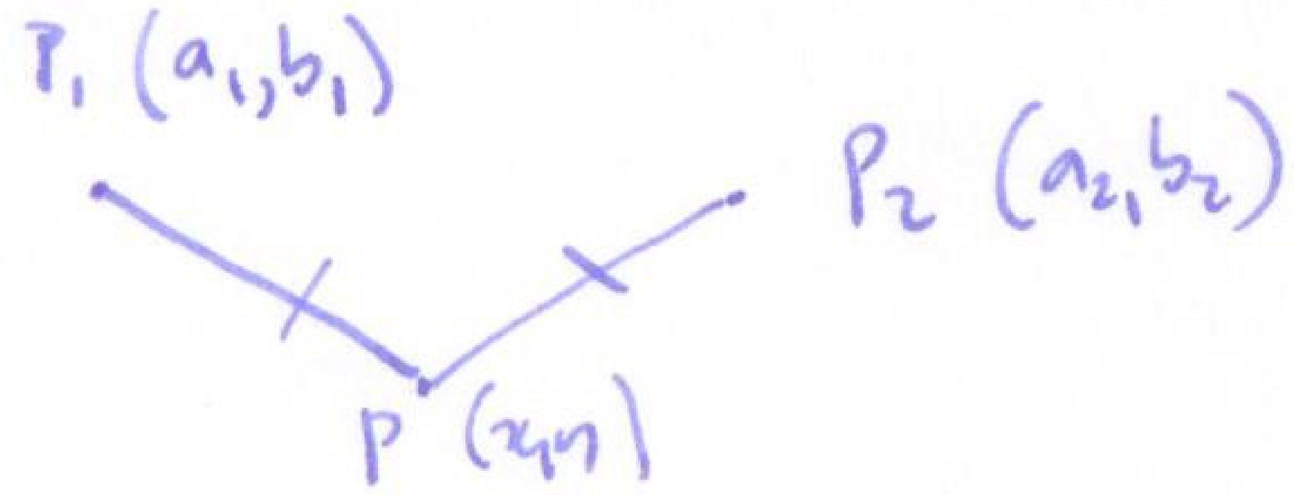
Equation of a circle:



$$r = \sqrt{(x-a)^2+(y-b)^2}$$

$$r^2 = (x-a)^2+(y-b)^2 \leftarrow \text{equation of a circle.}$$

Equidistant ~~line~~ set between two points (in fact a line)



same distance $\Rightarrow$

$$|PP_1|^2 = |PP_2|^2$$

$$(a_1-x)^2+(b_1-y)^2 = (a_2-x)^2+(b_2-y)^2$$

$$a_1^2-2a_1x+x^2+b_1^2-2b_1y+y^2 = a_2^2-2a_2x+x^2+b_2^2-2b_2y+y^2$$

typo on p. 52!

$$x\,2(a_2-a_1) + y\,2(b_2-b_1) + (a_1^2-a_2^2) + (b_1^2-b_2^2) = 0$$

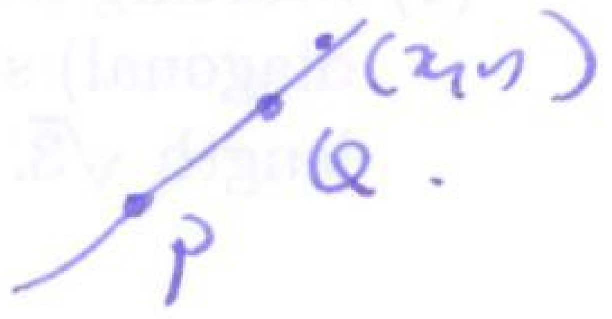
straight line!

§3.4 Intersections of lines and circles

correspondence:

	geometry		algebra
	construct a line/circle	$\Leftrightarrow$	find an equation
	construct intersections of lines/circles etc.	$\longleftrightarrow$	solving equations

examples:

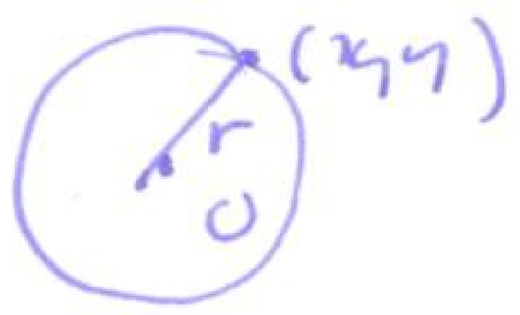


- draw a line thru $P(x_1,y_1)$ and $Q(x_2,y_2)$

slope of line: $\dfrac{y_2-y_1}{x_2-x_1}$ must pass through (x_1,y_1) say

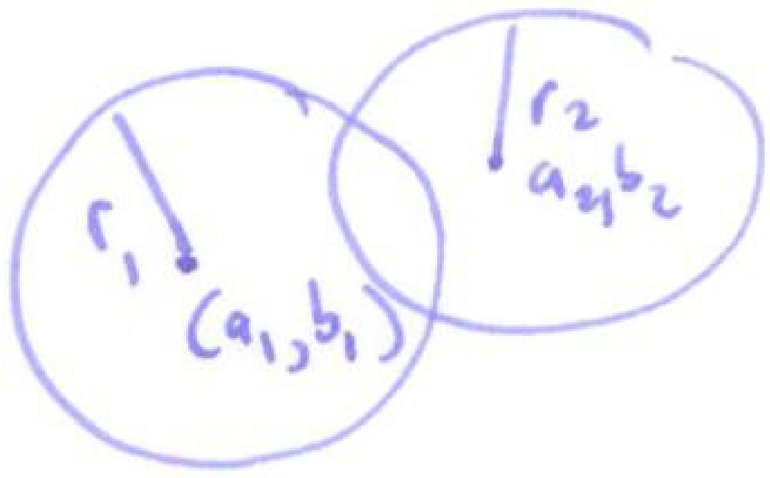
$$y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)$$

symmetric form: $(y-y_1)(x_2-x_1)=(y_2-y_1)(x-x_1)$



- draw a circle of radius R with center $O(a,b)$

$$(x-a)^2+(y-b)^2=r^2$$



- find intersections: solve a pair of equations.

two circles:

$$(x-a_1)^2+(y-b_1)^2=r_1^2$$
$$(x-a_2)^2+(y-b_2)^2=r_2^2$$

① $x^2-2a_1x+a_1^2+y^2-2yb_1+b_1^2=r_1^2$

② $x^2-2a_2x+a_2^2+y^2-2yb_2+b_2^2=r_2^2$

①-②: $2x(a_2-a_1)+2y(b_2-b_1)+b_1^2-b_2^2+r_2^2-r_1^2=0$.

solve for x ~~or~~ y and plug into ①: gives quadratic equation for x: $Ax^2+Bx+C=0$

with solutions $x=\dfrac{-B\pm\sqrt{B^2-4AC}}{2A}$

observation: solving linear equations just requires $+, -, \times, \div$

solving quadratic equations just requires " $\sqrt{\cdot}$.

this gives: (and we can do all of these with compass and straightedge!)

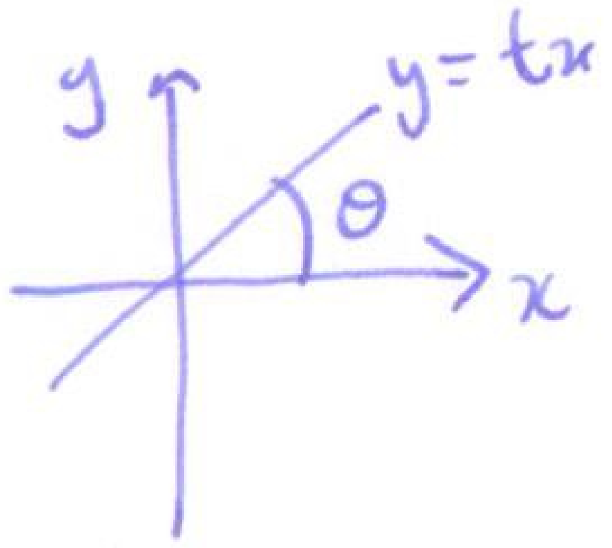
Algebraic criterion for constructibility A point is constructible (starting from the points $(0,0)$ and $(1,0)$) iff its coordinates are obtainable from 1 by the operations $+, -, \times, \div, \sqrt{\cdot}$.

Thm [Wantzel] (1800's) $\sqrt[3]{2}$, π, $\cos\left(\frac{\pi}{9}\right)$ not constructible.

so can't square the cube or circle, or trisect angles.

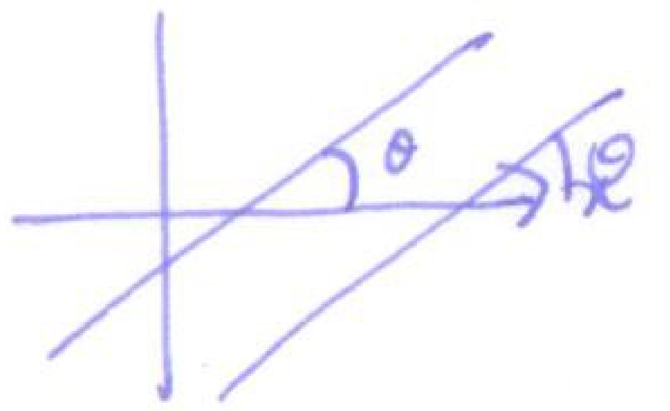
§3.5 Angle and slope

distance is an algebraic function of coordinates: $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.



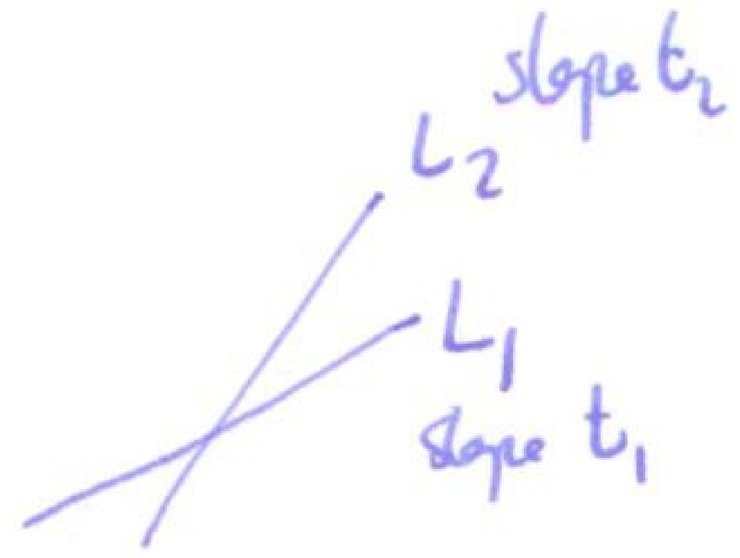
angle $\theta = \tan^{-1}(t) = \arctan(t)$ not an algebraic function.

however slope is an algebraic function, so often work with slope instead.



same slope $\Leftrightarrow$ same angle with x-axis.

what about angles between arbitrary lines?



Defn The slope of L_1 (with slope t_1) relative to the slope of L_2 (with slope t_2) is

$$\pm\left|\frac{t_1-t_2}{1+t_1t_2}\right|$$

note: • if $t_2 = 0$ just get t_1.

• $\pm|\cdot|$ slopes don't specify an angle! just a pair of angles that sum to π