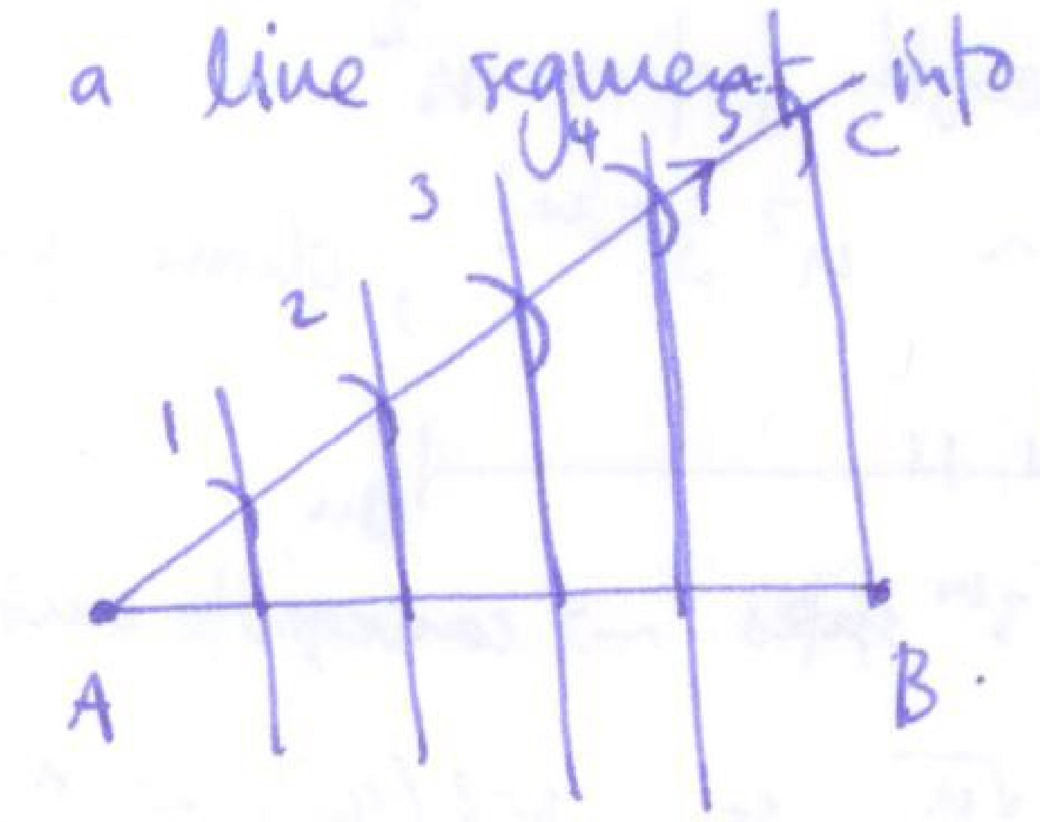
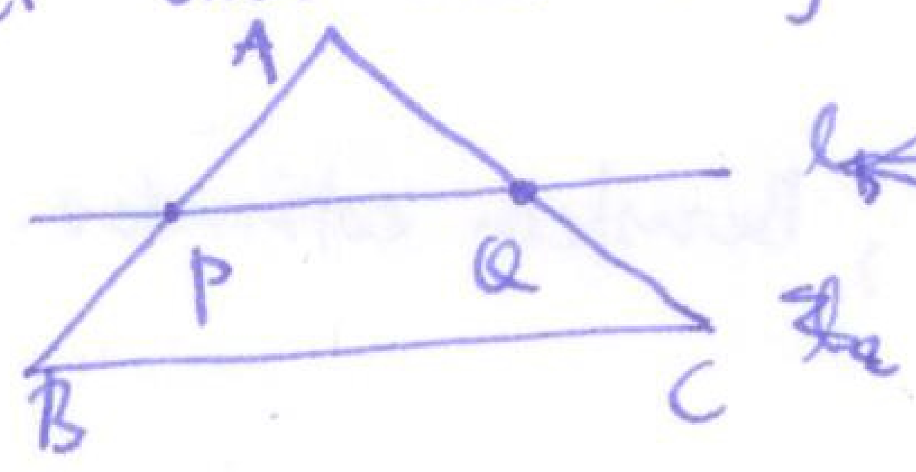
dividing line segments : easy   dividing angles : hard
(can't bisect an arbitrary angle)

we can divide a line segment into n parts:



construct parallel lines to CB.

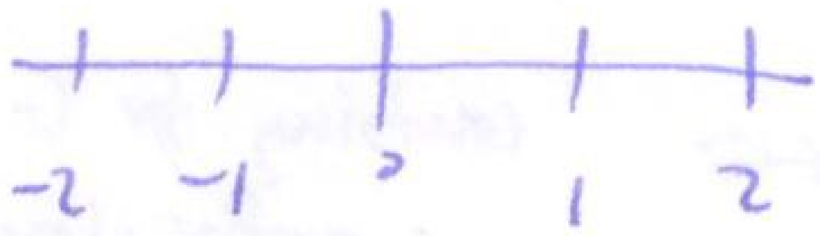
Thm (Thales) Parallel lines cut any lines they cross into proportional segments



← parallel line to BC

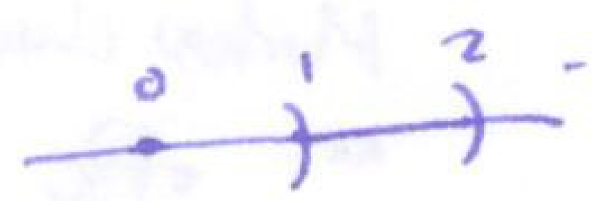
then $\frac{|AP|}{|PB|} = \frac{|AQ|}{|QC|}$

Remark : this means we can divide (and multiply lengths).

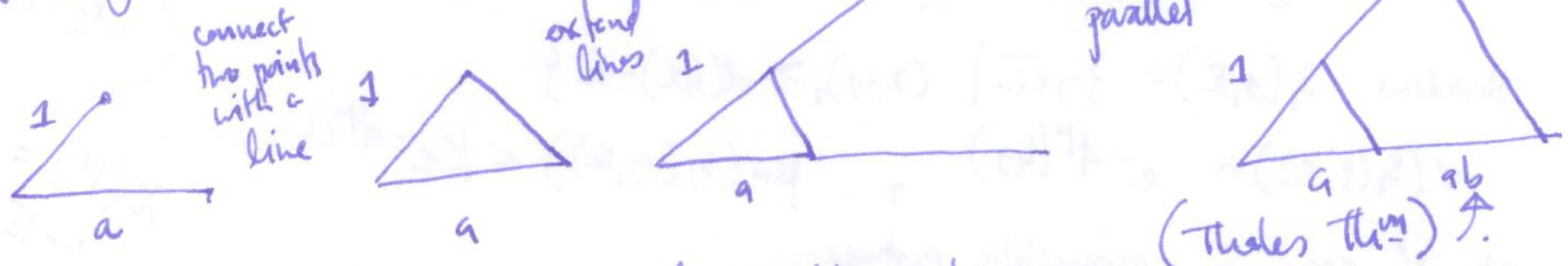
so we can construct  $\mathbb{Q}$ (what about $\sqrt{2}, \pi, e$?)

§1.4 Multiplication and division

choose some line segment to have length 1 

(we can construct lines of length n easily )

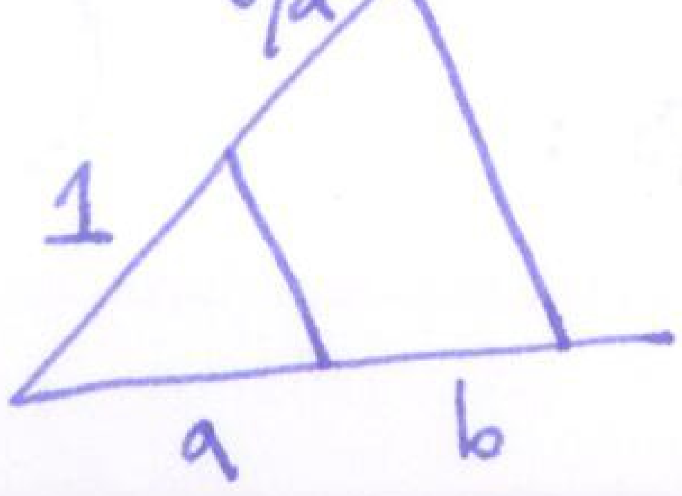
multiplication : given ~~another~~ line segments of lengths a, b construct a line segment of length ab



(Thales Thm)

division given line segments of length a, b construct a line segment of length a/b

similar :



so given lengths a, b we can construct $a+b$, $a-b$, ab, a/b

so we can construct any <u>rational</u> number ($\mathbb{Q}$).

<u>note</u>: some lengths are <u>not</u> rational, but can still be constructed

§1.5 similar triangles

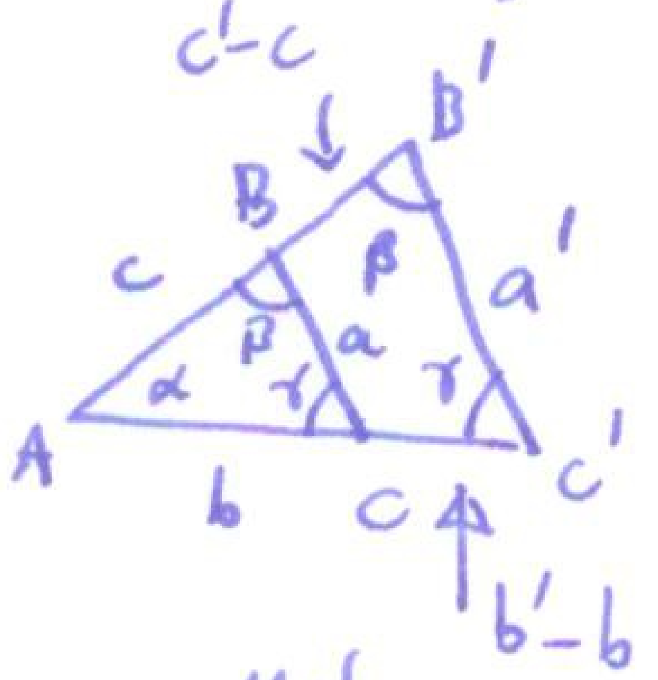
<u>note</u>: ancient greeks thought of products as <u>areas</u>.

<u>Def^n</u> Two triangles are <u>similar</u> if their corresponding angles are equal.

$$\frac{c}{c'} = \frac{a}{a'} = \frac{b}{b'}$$

<u>Claim</u> similar triangles have <u>proportional</u> sides, i.e. $\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|AC|}{|A'C'|}$

<u>Proof</u> move the two triangles so that:



angles β same so BC and $B'C'$ are parallel.

Thales: $\frac{b}{c} = \frac{b'-b}{c'-c}$

$$b(c'-c) = c(b'-b)$$

$$bc' - bc = cb' - bc$$

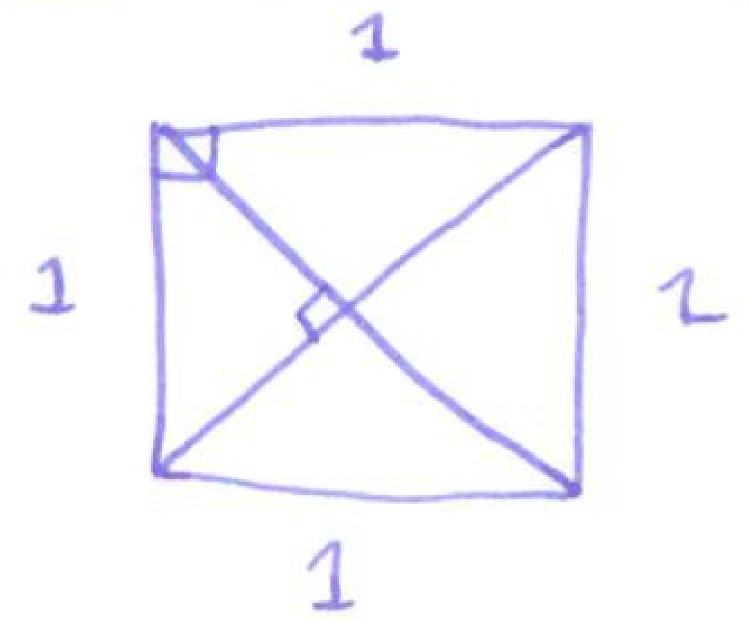
$$bc' = cb'$$

$$\frac{b}{b'} = \frac{c}{c'}$$

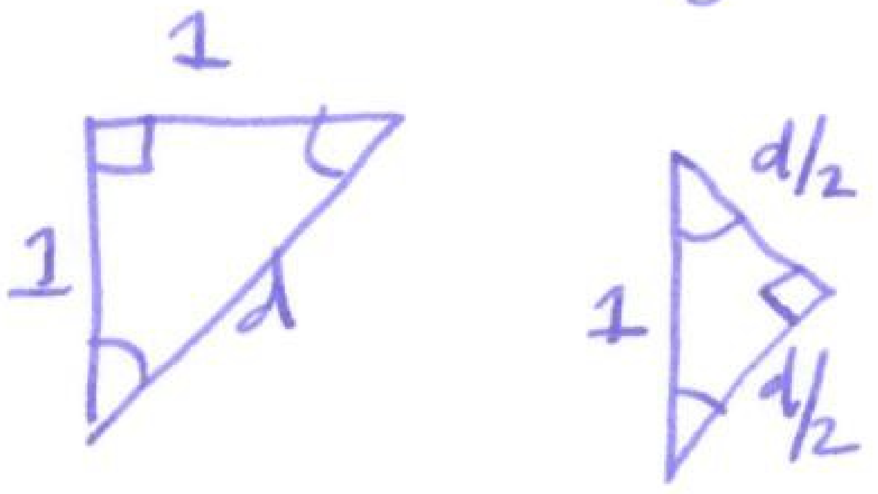
(now match angles β $\frac{a}{a'} = \frac{c}{c'}$

γ $\frac{a}{a'} = \frac{b'}{b'}$)

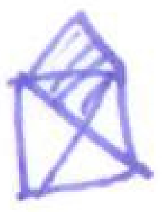
Corollary: diagonal of unit square has length $\sqrt{2}$.



similar triangles:



so $\dfrac{1}{d} = \dfrac{d/2}{1} \Rightarrow 2 = d^2 \qquad d = \sqrt{2}$.



Claim $\sqrt{2}$ is irrational (i.e. not a fraction $\frac{m}{n}$ $m,n \in \mathbb{N}$)

Proof (by contradiction)

suppose $\sqrt{2} = \dfrac{m}{n}$ $\quad m,n \in \mathbb{N}$, assume no common factor.

Then $2 = \dfrac{m^2}{n^2}$

$2n^2 = m^2 \Rightarrow m^2$ is even $\qquad (\text{even})^2 = \text{even}$, $(\text{odd})^2 = \text{odd}$.

$\Rightarrow m$ is even $\left(\begin{array}{l} (2a)^2 = 4a^2 \\ (2a+1)^2 = 4a^2+4a+1 \end{array} \right)$.

$\Rightarrow m = 2\ell$ for some $\ell \in \mathbb{N}$

$2n^2 = (2\ell)^2 = 4\ell^2$

$n^2 = 2\ell^2 \Rightarrow n^2$ is even

$\Rightarrow n$ is even # no common factor (both even) $\square$