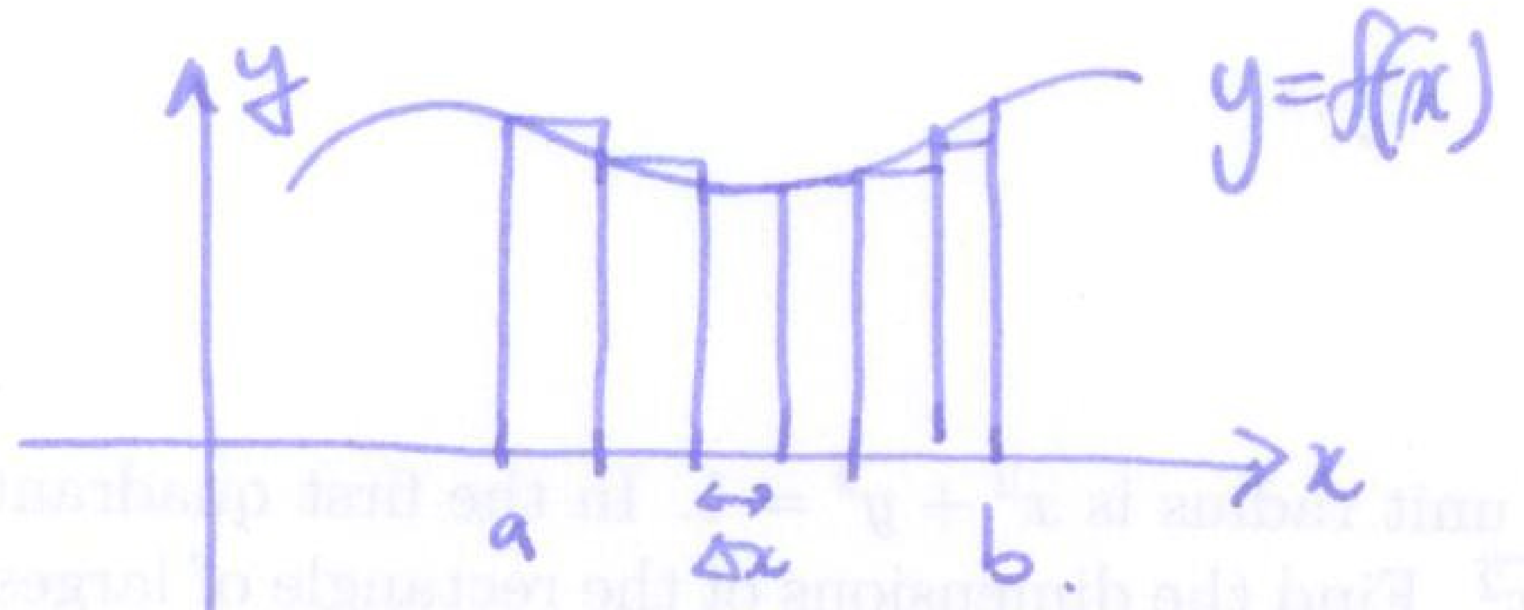
Notation



N rectangles so

$$\Delta x = \frac{b-a}{N}$$

left end point rectangles: $L_N = \Delta x \sum_{j=0}^{N-1} f(a + j\Delta x)$

right end points rectangles $R_N = \Delta x \sum_{j=1}^{N} f(a + j\Delta x)$

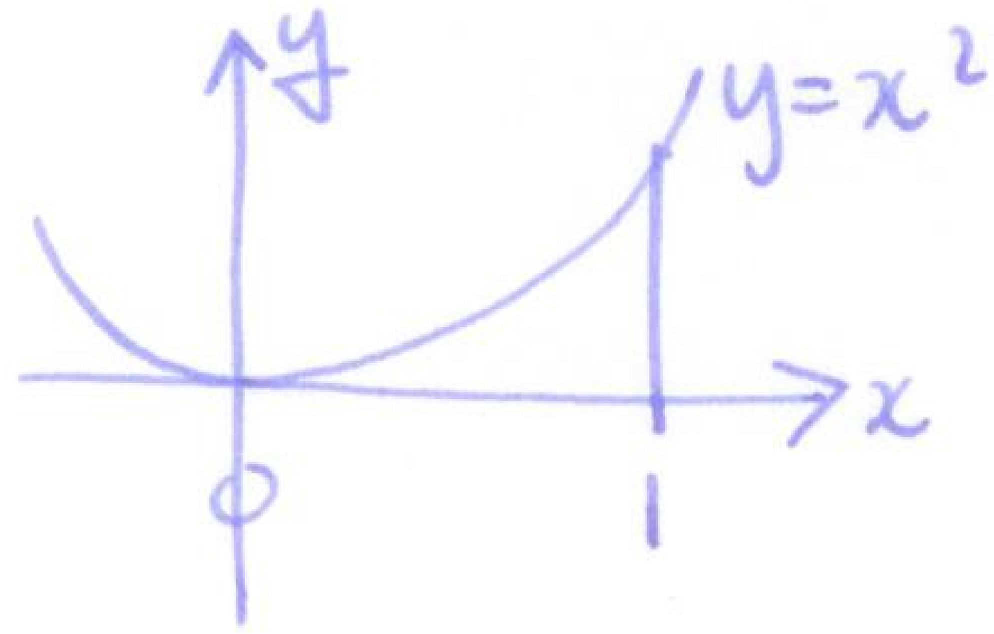
midpoint rectangles $M_N = \Delta x \sum_{j=1}^{N} f(a + (j-\frac{1}{2})\Delta x)$

key fact:

Thm If $f(x)$ is cts on $[a,b]$ then all three approximations limit to the area under the curve.

i.e. area under curve $= \lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N$

Example



n rectangles

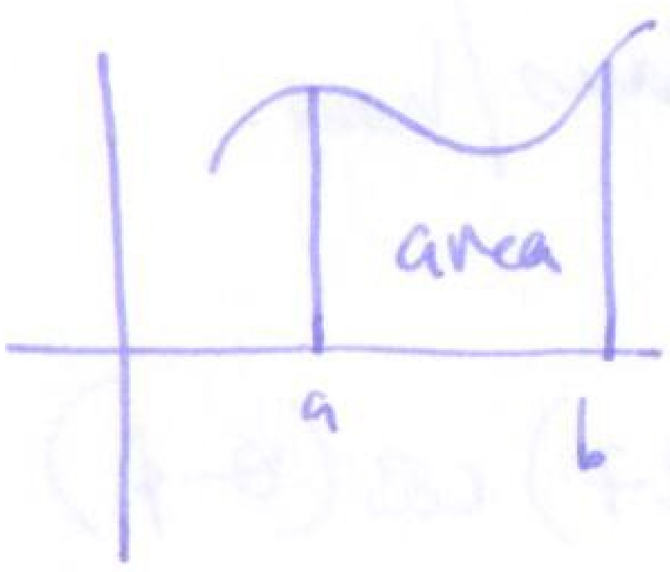
$$L_n = \sum_{k=0}^{n-1} \frac{1}{n} f\left(\frac{k}{n}\right) = \sum_{k=0}^{n-1} \frac{k^2}{n^3}$$

$$= \frac{1}{n^3} \sum_{k=0}^{n-1} k^2$$

note: $1^2 + 2^2 + \cdots + N^2 = \frac{1}{6} N(N+1)(2N+1)$

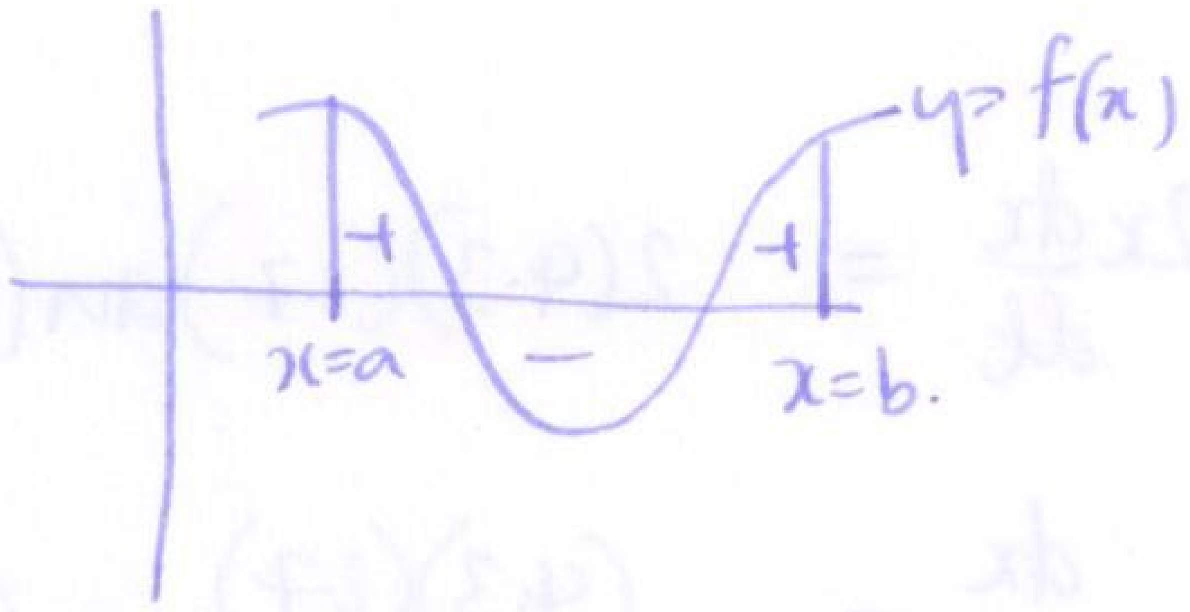
so area $= \lim_{n\to\infty} \frac{1}{n^3} \frac{1}{6} (n-1)(n)(2n-1) = \lim_{n\to\infty} \frac{1}{6} \frac{2n^3 + \cdots}{n^3} = \frac{1}{3}$.

§5.2 Definite integral

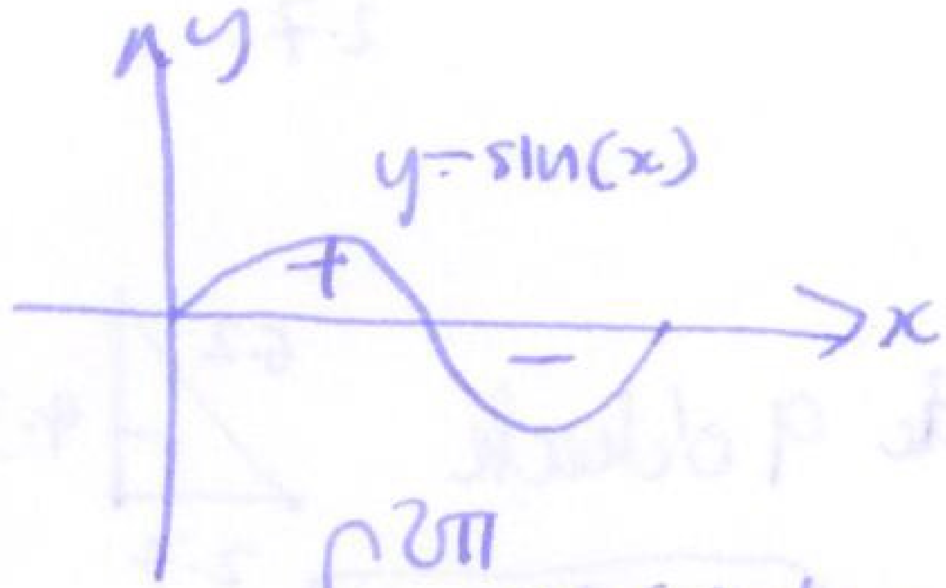


$$\int_a^b f(x)\,dx = \text{signed area under curve } y=f(x) \text{ between } x=a \text{ and } x=b$$

signed area:



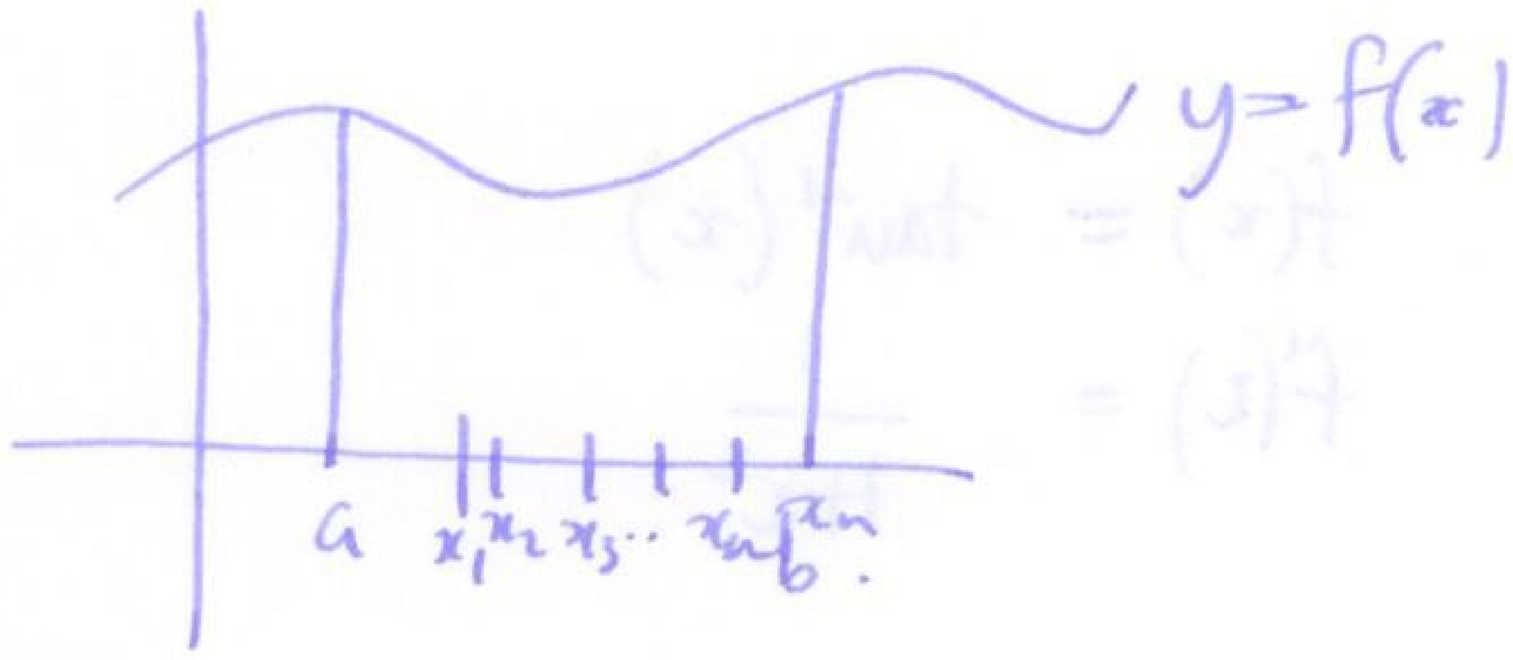
so



$$\int_0^{2\pi} \sin(x)\,dx = 0$$

Formal definition

Riemann sum $R(f, P, C)$



P: partition of $[a,b]$ i.e. a sequence of points $a = x_0 \leq x_1 < x_2 \cdots x_n = b$

C: choice of points $c_i \in [x_{i-1}, x_i]$

$$R(f,P,C) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i \quad \text{where } \Delta x_i = x_i - x_{i-1}.$$

set $\|P\| = \max \Delta_i$

<u>Defn</u> $$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} R(f,P,C) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

when this limit exists we say f is <u>integrable</u> over $[a,b]$

useful properties

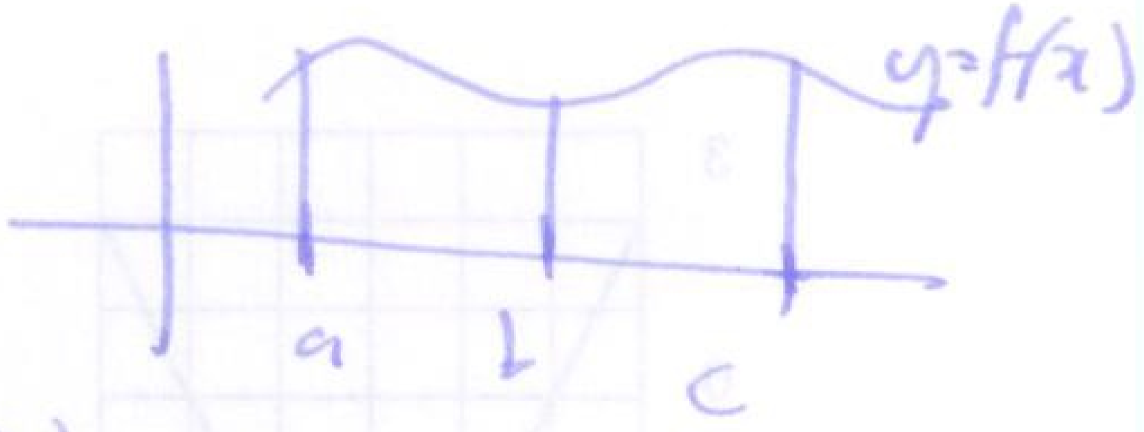
$$\int_a^b c\,dx = c(b-a)$$

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

0-length integral: $\int_a^a f(x)\,dx = 0$



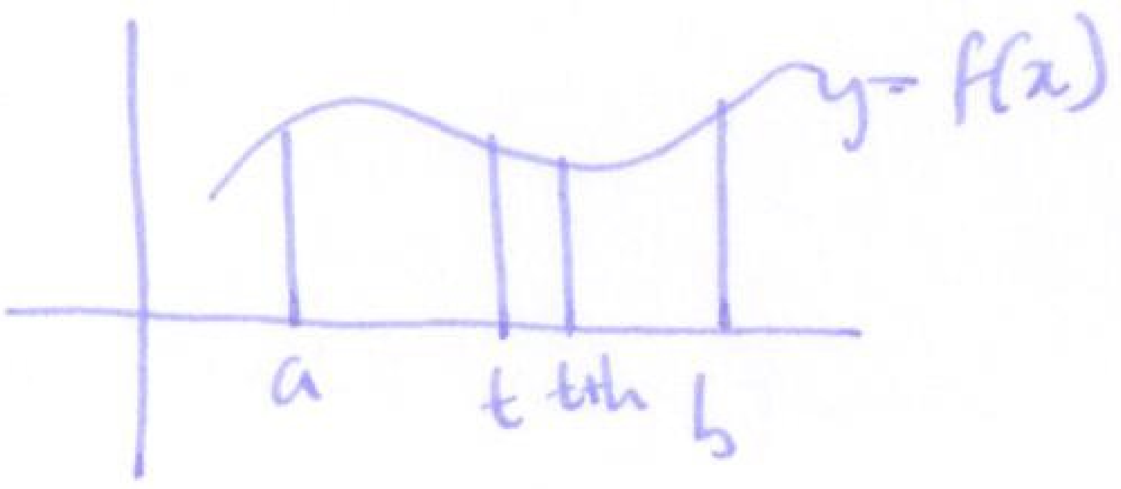
adjacent intervals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

comparison: if $f(x) \leq g(x)$ on $[a,b]$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

§5.3 Fundamental theorem of calculus

Thm FTC(1) suppose $f(x)$ is cts on $[a,b]$ and $F(x)$ is an antiderivative for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$

intuition:



consider $\int_a^t f(x)\,dx = F(t) - F(a)$

Q: what is rate of change wrt t?

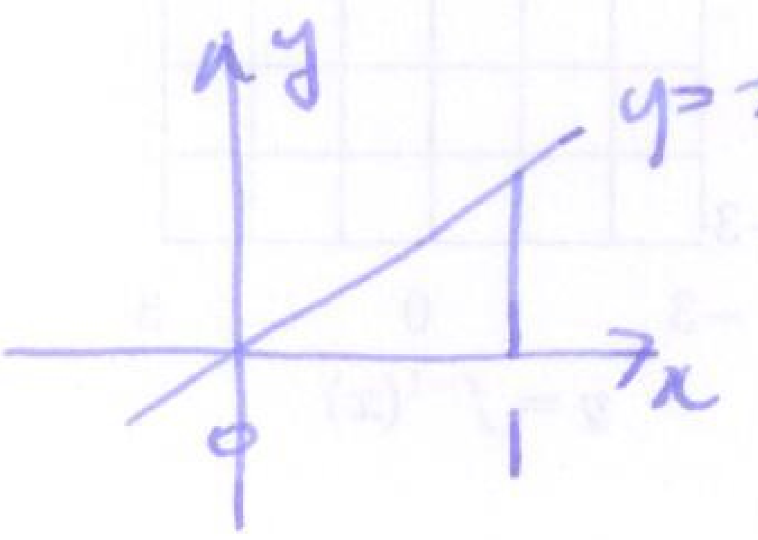
LHS: $\frac{d}{dt}\left(F(t)-F(a)\right) = F'(t)$

varying $t \leftrightarrow$ adding an a thin rectangle of area $\approx f(t)h$

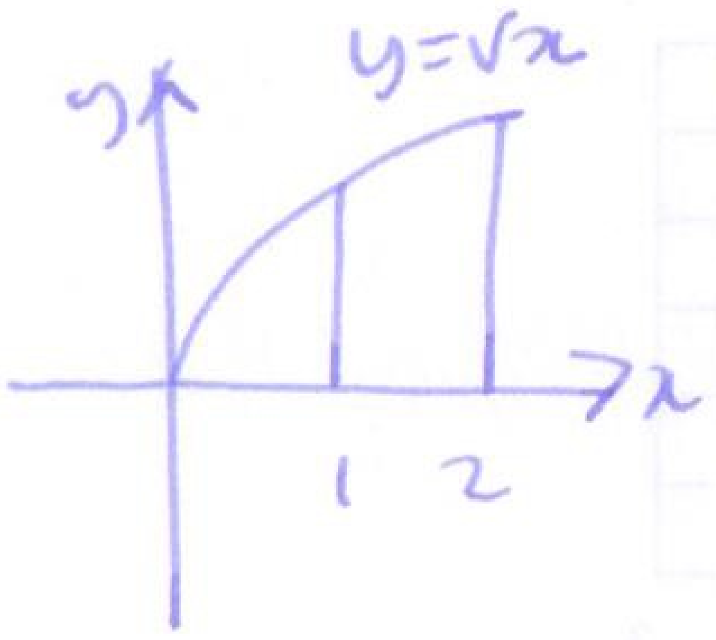
$$\lim_{h\to 0} \frac{F(t+h)-F(t)}{h} \approx \lim_{h\to 0} \frac{F(t)+hf(t)-F(t)}{h} = f(t).$$

so $f(t) = F'(t)$, i.e. $F(t)$ is an antiderivative for $f(t)$.

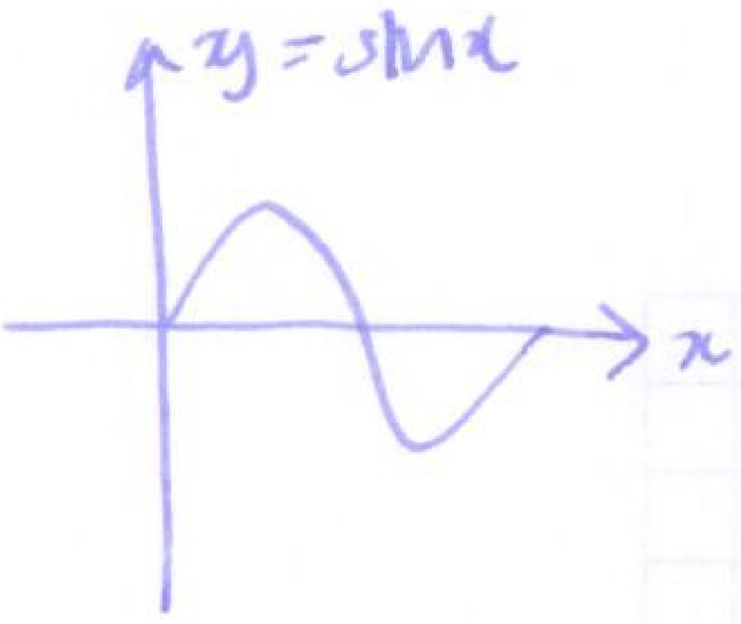
Examples



$$\int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$

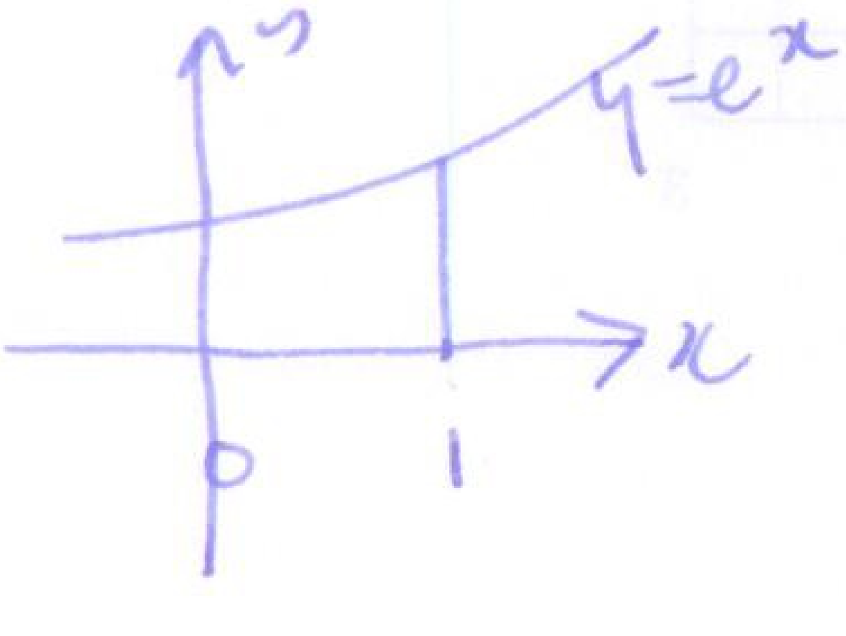


$$\int_1^2 x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_1^2 = \tfrac{2}{3}\left(2^{3/2}-1\right)$$

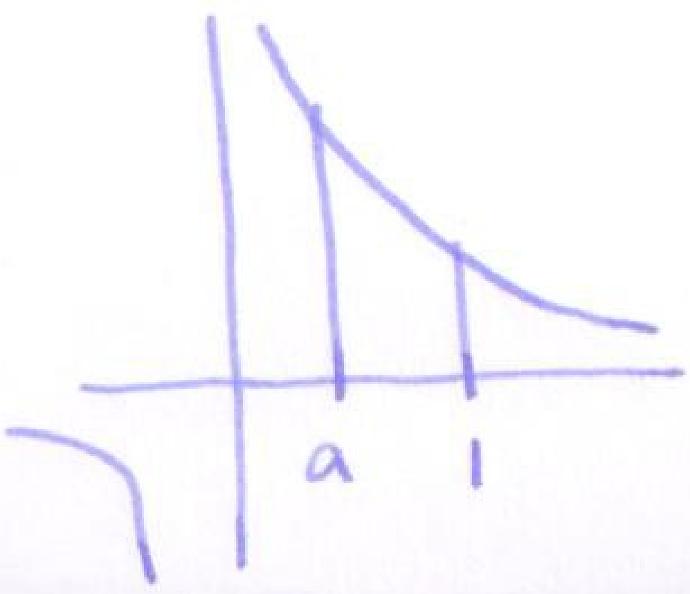


$$\int_0^{\pi} \sin(x)\,dx = \left[-\cos x\right]_0^{\pi} = -\cos(\pi) + \sin(c)$$

$$= -(-)+1 = 2.$$



$$\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1$$



$$\int_a^1 \frac{1}{x}\,dx = \left[\ln|x|\right]_a^1 = \ln(1) - \ln(a)$$

$$= -\ln(a)$$

$\to \infty$ as $a \to 0$.