Notation indefinite integral

$$\int f(x)\,dx = F(x) + c$$

means : $F(x)+c$ is the general anti-derivative for $f(x)$.

Thm $$\int x^n\,dx = \frac{1}{n+1} x^{n+1} + c \quad \text{for } n \neq -1$$

Proof : $$\frac{d}{dx}\left(\frac{1}{n+1} x^{n+1} + c\right) = x^n \quad \square$$

Thm $$\int \frac{1}{x}\,dx = \ln|x| + c$$

Proof $$\frac{d}{dx}\left(\ln(x) + c\right) = \frac{1}{x} \quad (x>0)$$

$$\frac{d}{dx}\left(\ln(-x) + c\right) = -\frac{1}{-x} = \frac{1}{x} \quad (x<0) \quad \square$$

Thm sums and constant multiples

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

Warning : no product / quotient / chain rule for integrals!

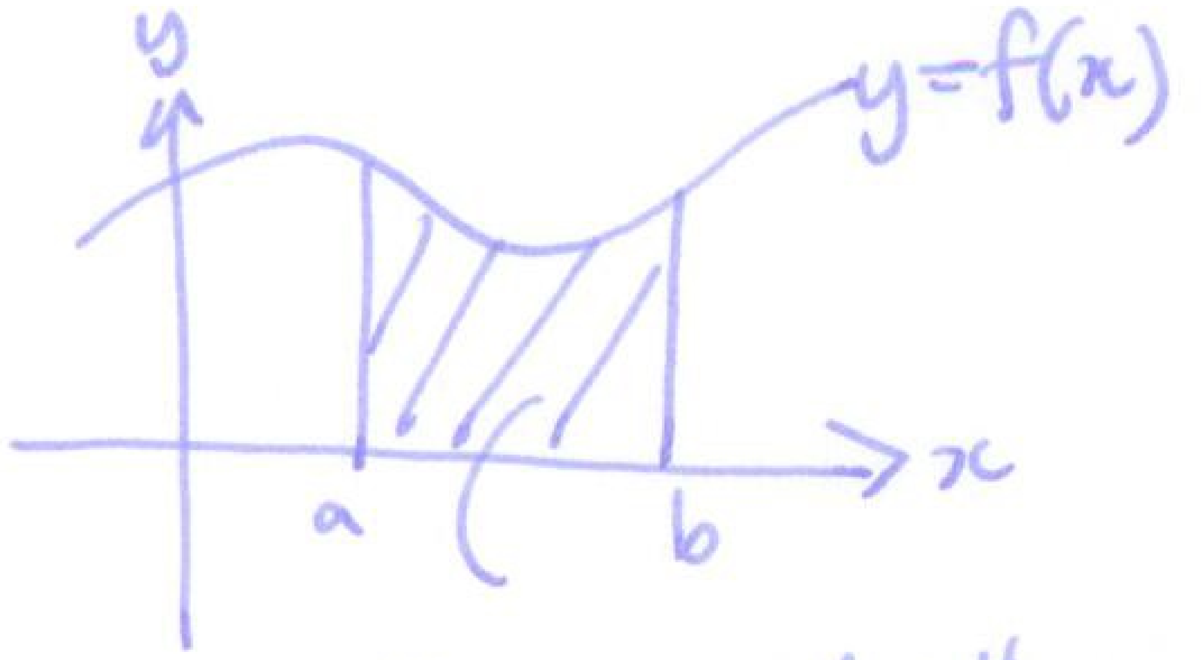
Useful integrals

$$\int \sin(x)\,dx = -\cos(x) + c \qquad \int e^x\,dx = e^x + c$$

$$\int \cos(x)\,dx = \sin(x) + c$$

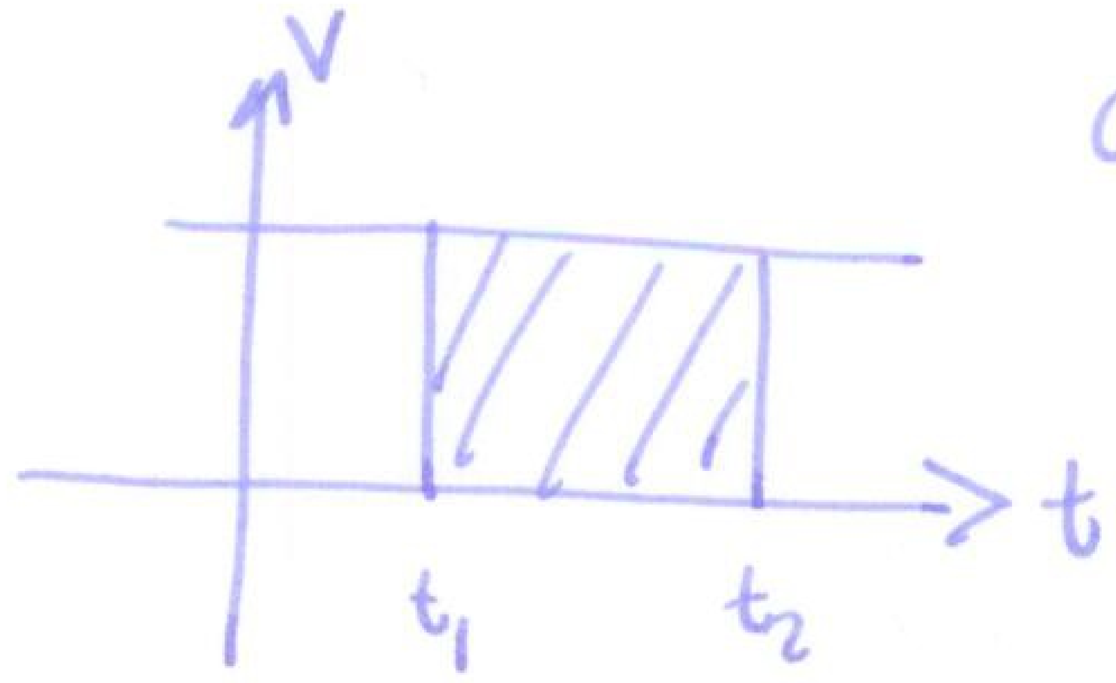
Example $\int x^2 + \frac{1}{x} + \sin(x)\, dx$

§5.1 Approximating areas



area under the graph between $x=a$ and $x=b$

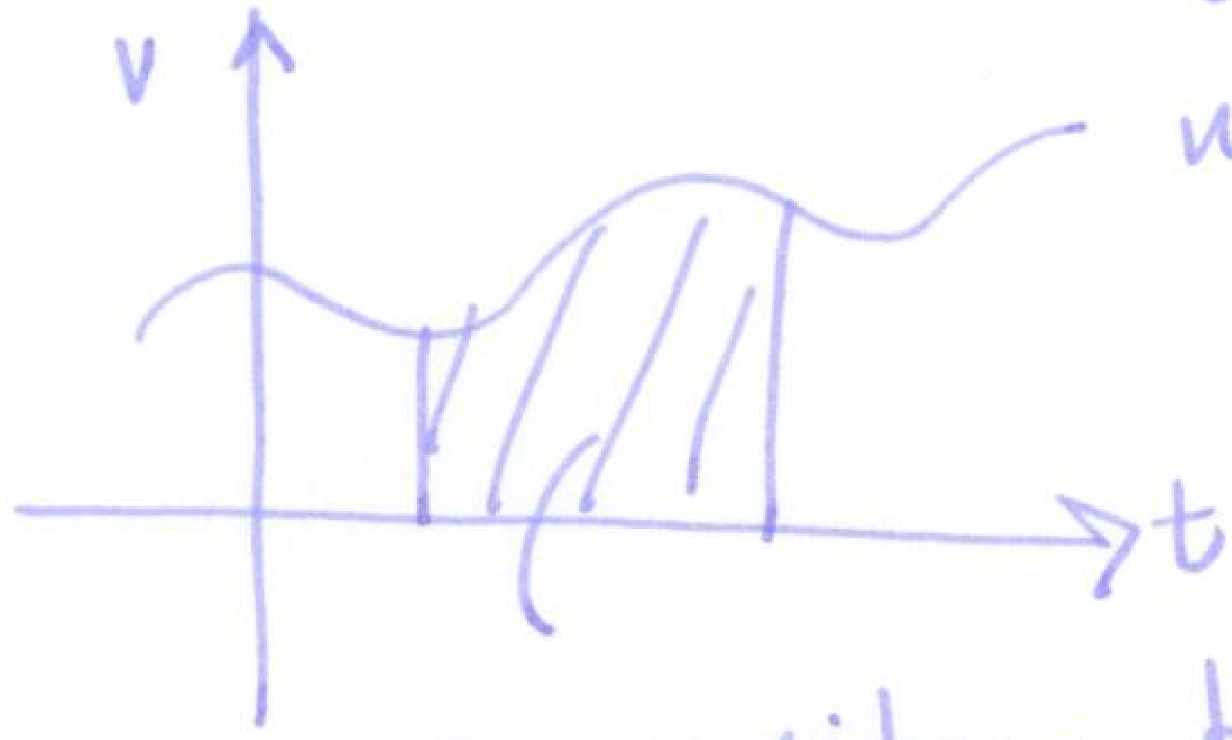
e.g.



constant velocity

distance travelled = velocity × time

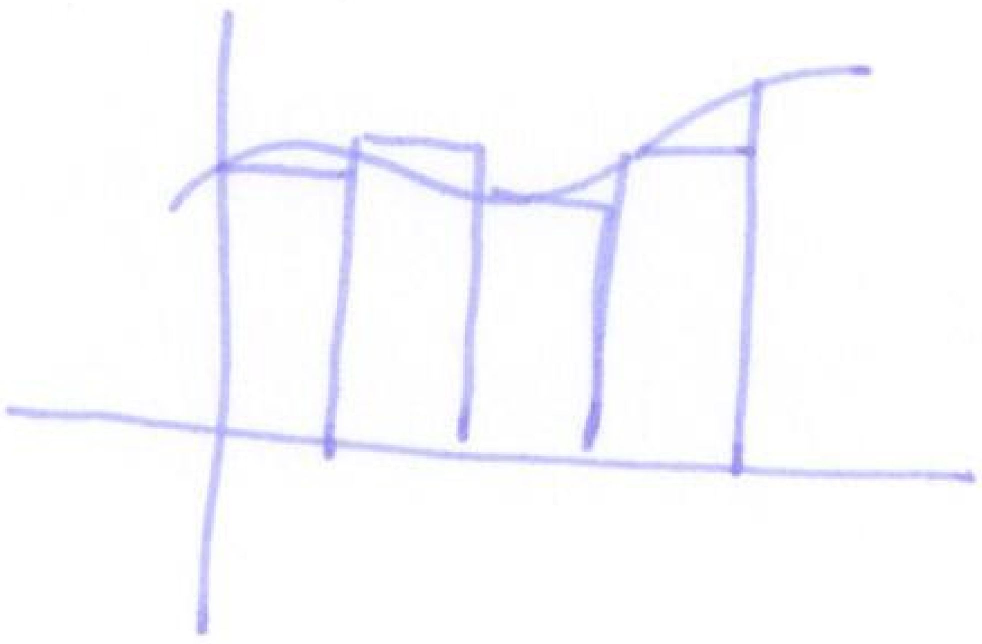
= area under graph



non-constant velocity

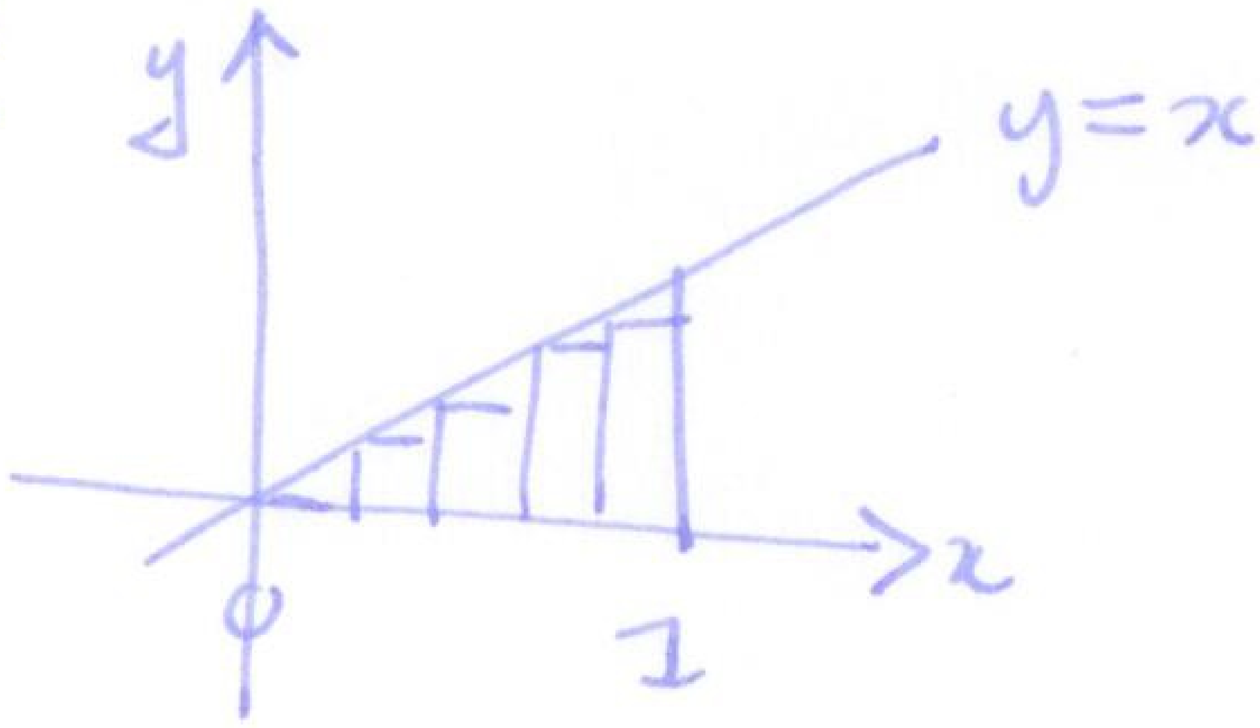
area = distance travelled

finding the area:



approximate by rectangles

Example



find area under $y=x$ between 0 and 1 (answer = ½)

$$\sum \text{area of rectangles} = \sum f(x).\text{width}.$$

five rectangles: $f(0)\cdot\frac{1}{5} + f(\frac{1}{5})\frac{1}{5} + f(\frac{2}{5})\frac{1}{5} + f(\frac{3}{5})\frac{1}{5} + f(\frac{4}{5})\frac{1}{5}$

$$= \frac{1}{5}\left(0 + \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right) = \frac{1}{25}(10) = \frac{10}{25}$$

n rectangles: $f(0)\frac{1}{n} + f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \dots + f\left(\frac{n-1}{n}\right)\frac{1}{n}$.

$$= \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$$

in this example $f(x) = x$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\frac{k}{n} = \frac{1}{n^2}\sum_{k=0}^{n-1} k$$

<u>note</u> if $S_N = 1 + 2 + 3 + \dots + N$ then $S_n = \frac{1}{2}N(N+1)$

<u>Proof</u> ① induction: suppose true for $N = k$: $S_k = 1+2+3+\dots+k = \frac{1}{2}k(k+1)$
what about S_{k+1}?

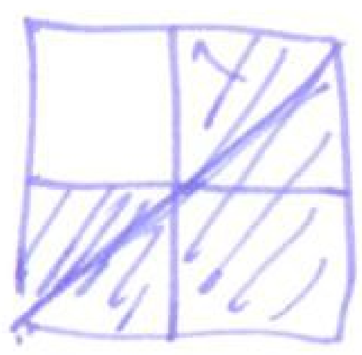
$$S_{k+1} = \underbrace{1 + 2 + \dots + k}_{S_k} + k + 1$$

$$= \frac{1}{2}k(k+1) + k + 1 = (k+1)\left(\frac{1}{2}k + 1\right) = \frac{1}{2}(k+1)(k+2)$$

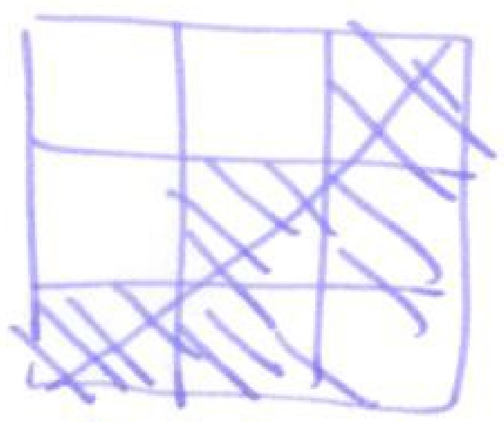
$$= S_{k+1}.$$

so true for $k \Rightarrow$ true for $k+1$ □.

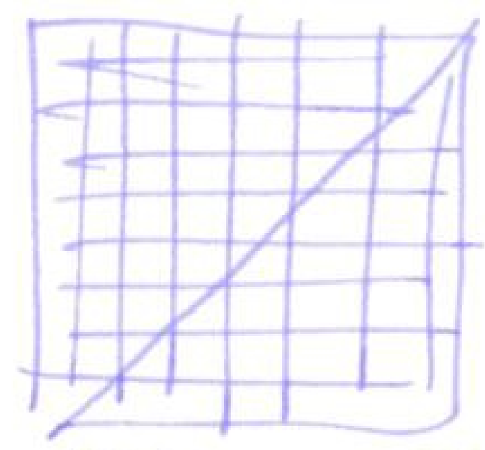
②



$1 + 2 = \frac{1}{2}(2)^2 + \frac{1}{2}2$



$1 + 2 + 3 = \frac{1}{2}(3)^2 + \frac{1}{2}3.$



$1+1+\dots+n$

$= \frac{1}{2}(n)^2 + \frac{1}{2}n$

$= \frac{1}{2}n(n+1).$

in our example:

$$\text{area} \approx \frac{1}{n^2}\sum_{k=0}^{n-1} k = \frac{1}{n^2}\frac{1}{2}(n-1)n = \frac{(n-1)}{2n}$$

$$\lim_{n\to\infty}\frac{n-1}{2n} = \frac{1}{2} = \text{correct answer.}$$