(2) $\lim\limits_{x\to\frac{\pi}{2}} \frac{\cos^2 x}{\sin x - 1} = \lim\limits_{x\to\frac{\pi}{2}} \frac{2\cos x \cdot -\sin x}{\cos x} = \lim\limits_{x\to\frac{\pi}{2}} -2\sin(x) = -2$

(3) $\lim\limits_{x\to 0+} x\ln(x) = \lim\limits_{x\to 0+} \frac{\ln(x)}{1/x} = \lim\limits_{x\to 0+} \frac{1/x}{-x^{-2}} = \lim\limits_{x\to 0} (-x) = 0$

(4) $\lim\limits_{x\to 0} \frac{e^x - x - 1}{\cos(x) - 1} = \lim\limits_{x\to 0} \frac{e^x - 1}{-\sin x} = \lim\limits_{x\to 0} \frac{e^x}{-\cos(x)} = -e.$

<u>Warning</u>: $\lim\limits_{x\to 0} \frac{x^2+1}{x+1} = 1$! not indeterminate form
can't use L'Hôpital

(5) $\lim\limits_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim\limits_{x\to 0} \frac{x - \sin x}{x\sin x} = \lim\limits_{x\to x} \frac{1 + \cos x}{\sin x + x\cos x}$

$= \lim\limits_{x\to 0} \frac{-\sin x}{\cos x + \cos x + x\cdot -\sin x} = \lim\limits_{x\to 0} \frac{-\sin x}{2\cos x \mp x\sin x} = 0$

(6) $\lim\limits_{x\to 0+} x^x = \lim\limits_{x\to 0+} e^{x\ln(x)}$ $\quad e^x$ continuous

so $\lim\limits_{x\to 0+} e^{x\ln(x)} = e^{\lim\limits_{x\to 0+} x\ln x}$

so find $\lim\limits_{x\to 0+} x\ln x = 0$ so $\lim\limits_{x\to 0+} e^{x\ln x} = e^0 = 1.$

<u>Comparing growth of functions</u>

<u>Q</u>: which grows faster $(\ln(x))^2$ or $\sqrt{x}$?

$\lim\limits_{x\to\infty} \frac{\sqrt{x}}{(\ln(x))^2} = \lim\limits_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{2(\ln(x))\cdot\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{x^{1/2}}{4\ln(x)} = \lim\limits_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{4/x}$

$= \lim\limits_{x\to\infty} \frac{1}{8}\sqrt{x} = \infty.$

Thm e^x grows faster than ~~any polynomial~~ x^n for any n.

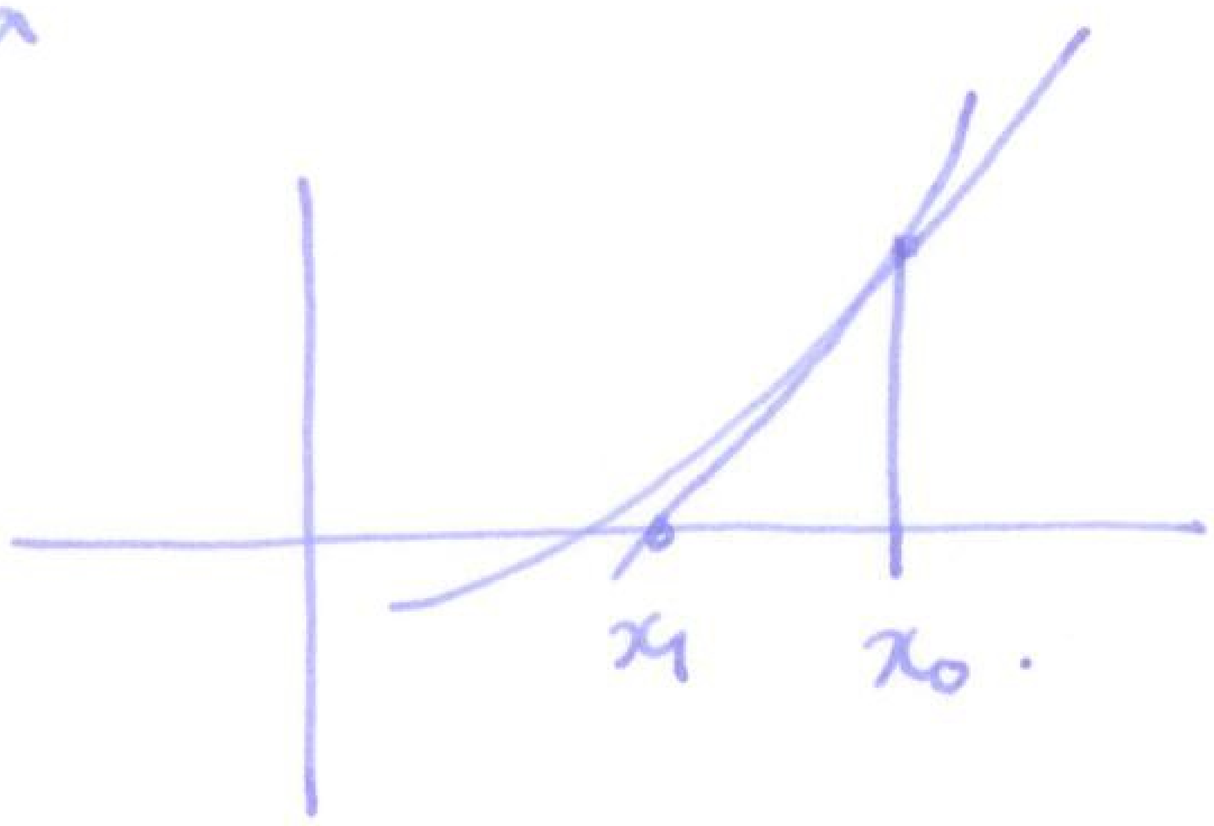
Proof $\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$.

§4.8 Newton's method

Q: find roots of $x^5 - x - 1$ (one real root)

problem: no formula

Newton's method:



guess initial value x_0

use linear approximation to find better approximation x_1

recall: equation of tangent line: $y - y_0 = m(x - x_0)$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

set $y = 0$ and solve for x: $0 - f(x_0) = f'(x_0)(x - x_0)$

$$\frac{-f(x_0)}{f'(x_0)} = x - x_0.$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

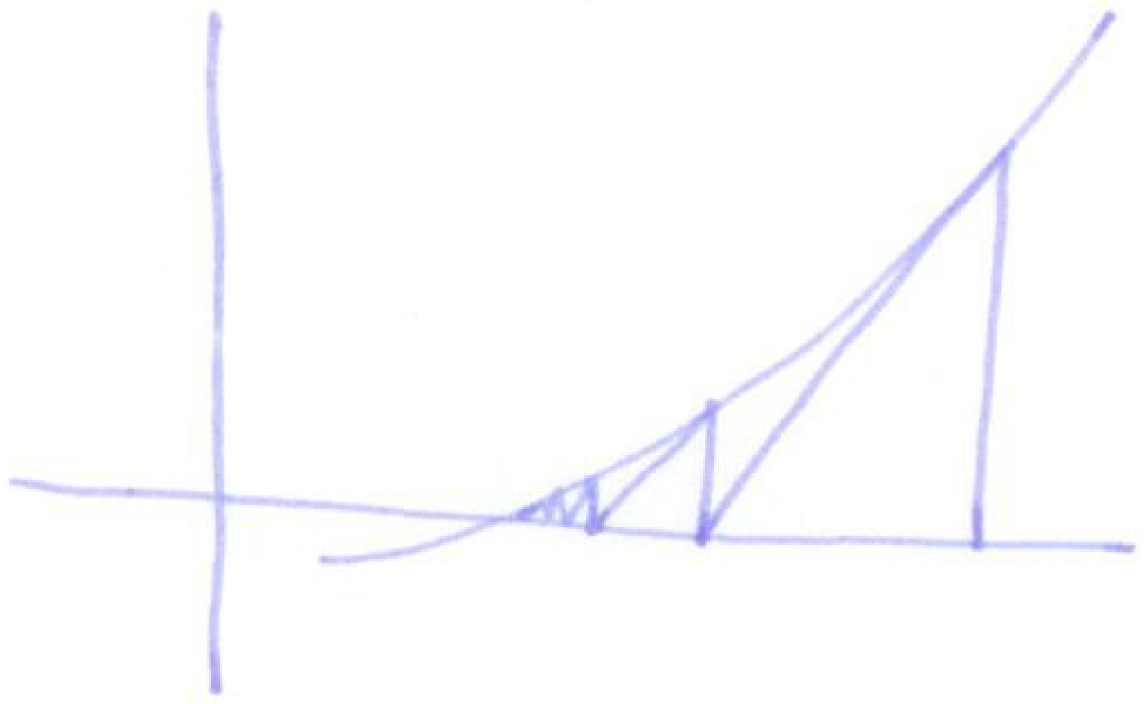
repeat: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ .... $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example $f(x) = x^5 - x - 1$ $\quad x_0 = 1$

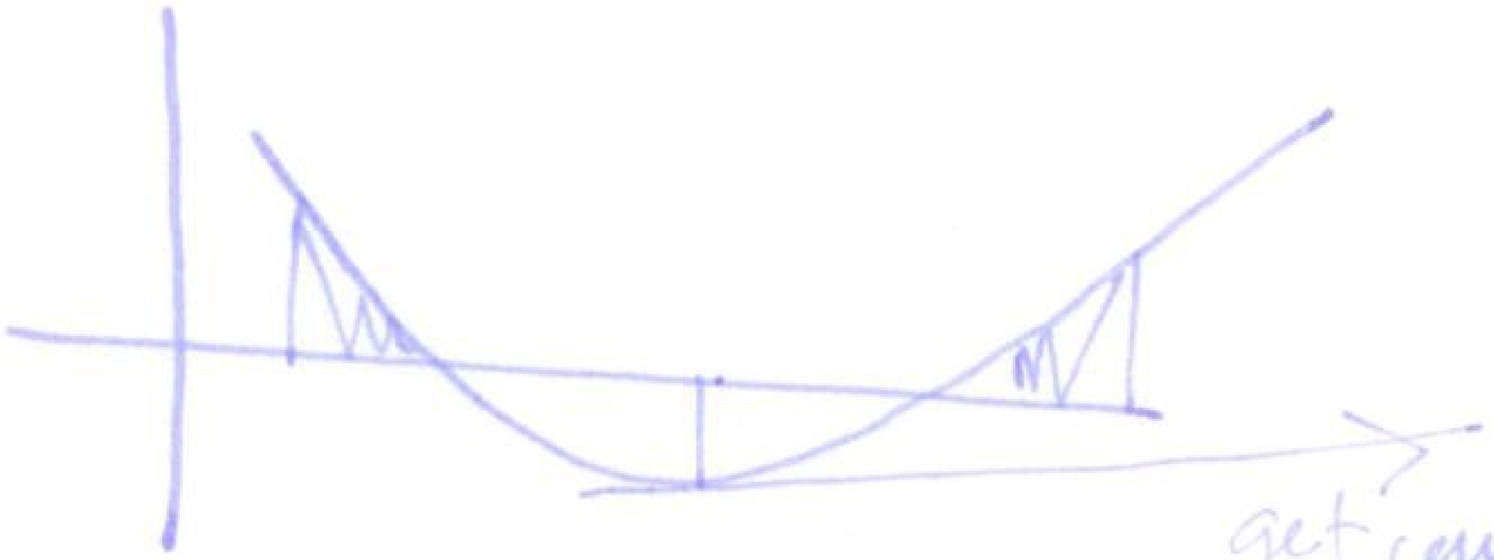
$f'(x) = 5x^4 - 1$ $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{4} = 1.25$

$x_2 = 1.25 - \frac{f(1.25)}{f'(1.25)} = \ldots$ and so on.

picture

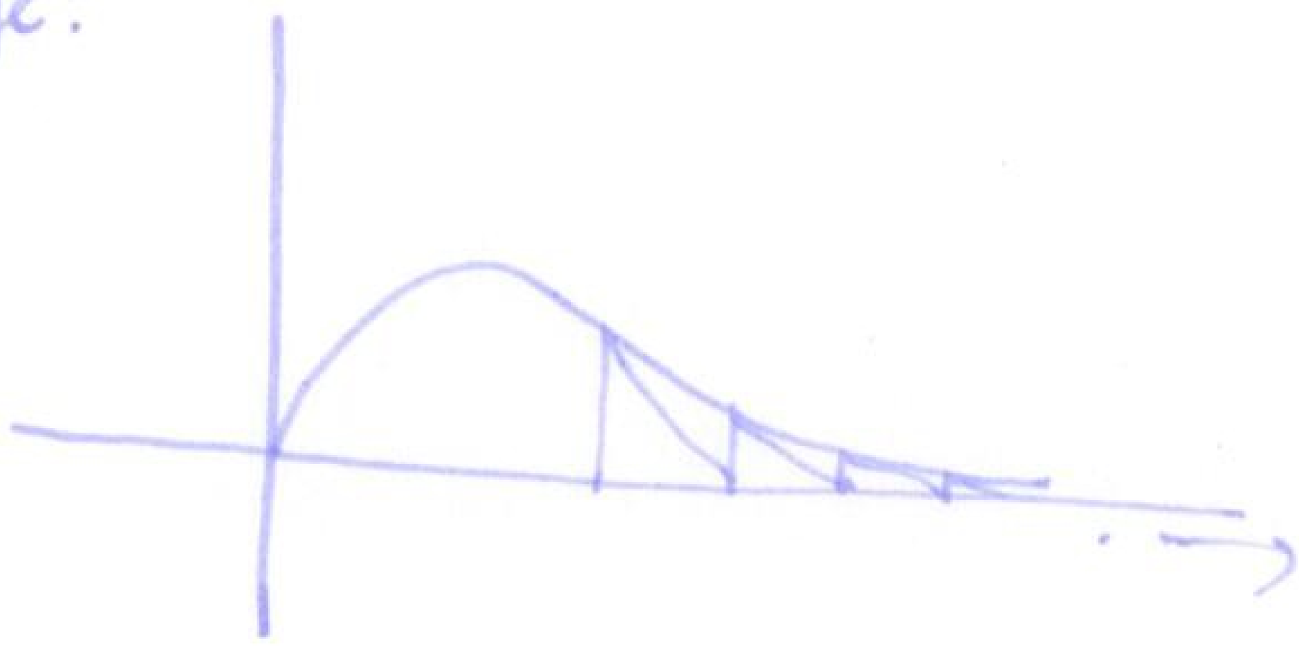


problems may be many roots:



get completely different root?

may not converge:



§4.9 Antiderivatives

Defn $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

Example $f(x) = x^2$ then $F(x) = \frac{1}{3}x^3$ is an antideriative

as $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$. **Note** $F(x) = \frac{1}{3}x^3 + 4$ also works.

General anti-derivative

Thm Let $F(x)$ be an antiderivative for $f(x)$. Then any other antiderivative for $f(x)$ is of the form $F(x)+c$ for some constant c.

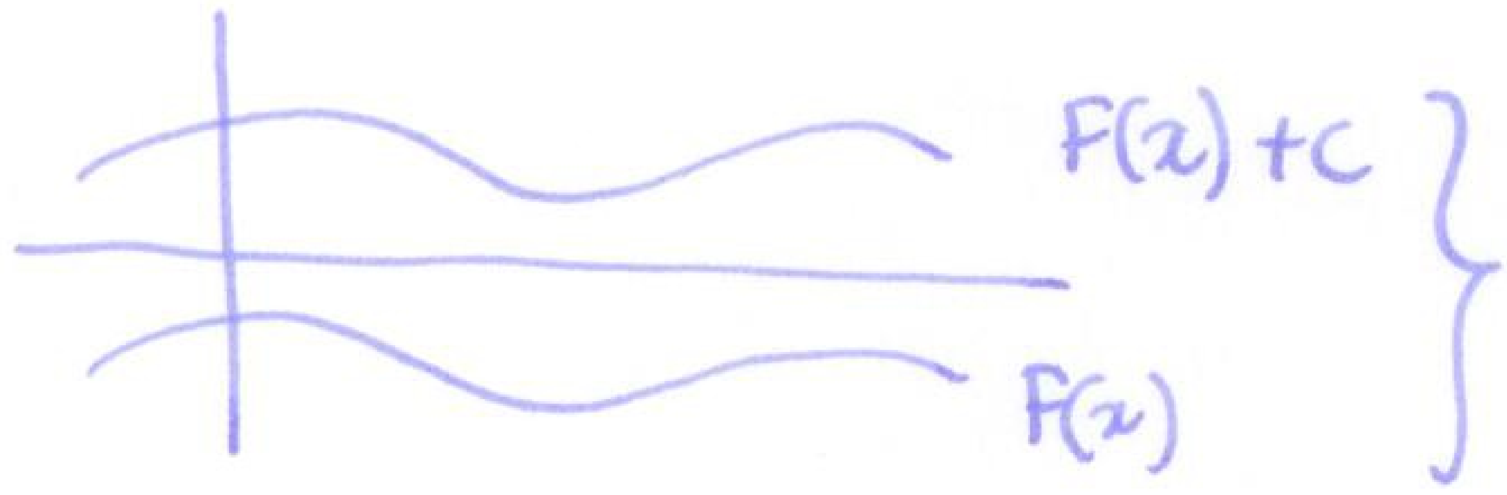
Proof Suppose $F(x)$ and $G(x)$ are anti-derivatives for $f(x)$,

~~Then any other anti-derivative so~~

i.e. $F'(x) = f(x)$ and $G'(x) = f(x)$. Then $F(x)-G(x)$ has derivative $\big(F(x)-G(x)\big)' = F'(x) - G'(x) = f(x) - g(x) = 0$

so $F(x) - G(x) =$ constant $\square$.

Picture $f(x)$ determines the slope of the anti-derivative.



same slope!

Example find the general anti-derivative to $f(x) = \sin 4x$

guess: $\frac{d}{dx}\big(\cos(4x)\big) = -4\sin(4x)$

so try: $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x)\right) = -\frac{1}{4}\cdot -4\sin(4x) = \sin 4x$.

so general antiderivative is $F(x) = -\frac{1}{4}\cos(4x) + C$.