

Example $f(x) = \frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}$

(71)

vertical asymptotes at $x = \pm 1$

horizontal asymptote at $y = 0$.

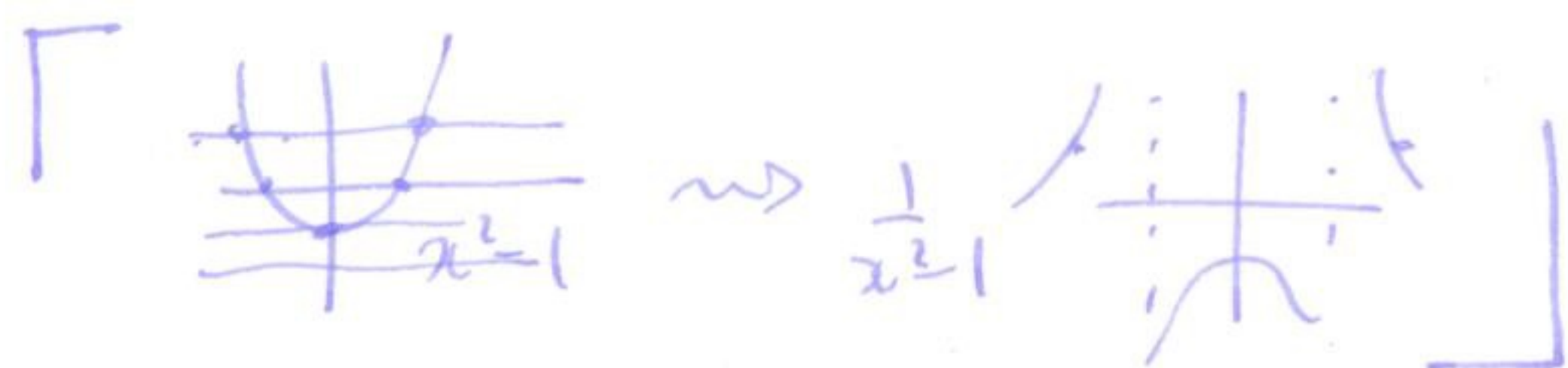
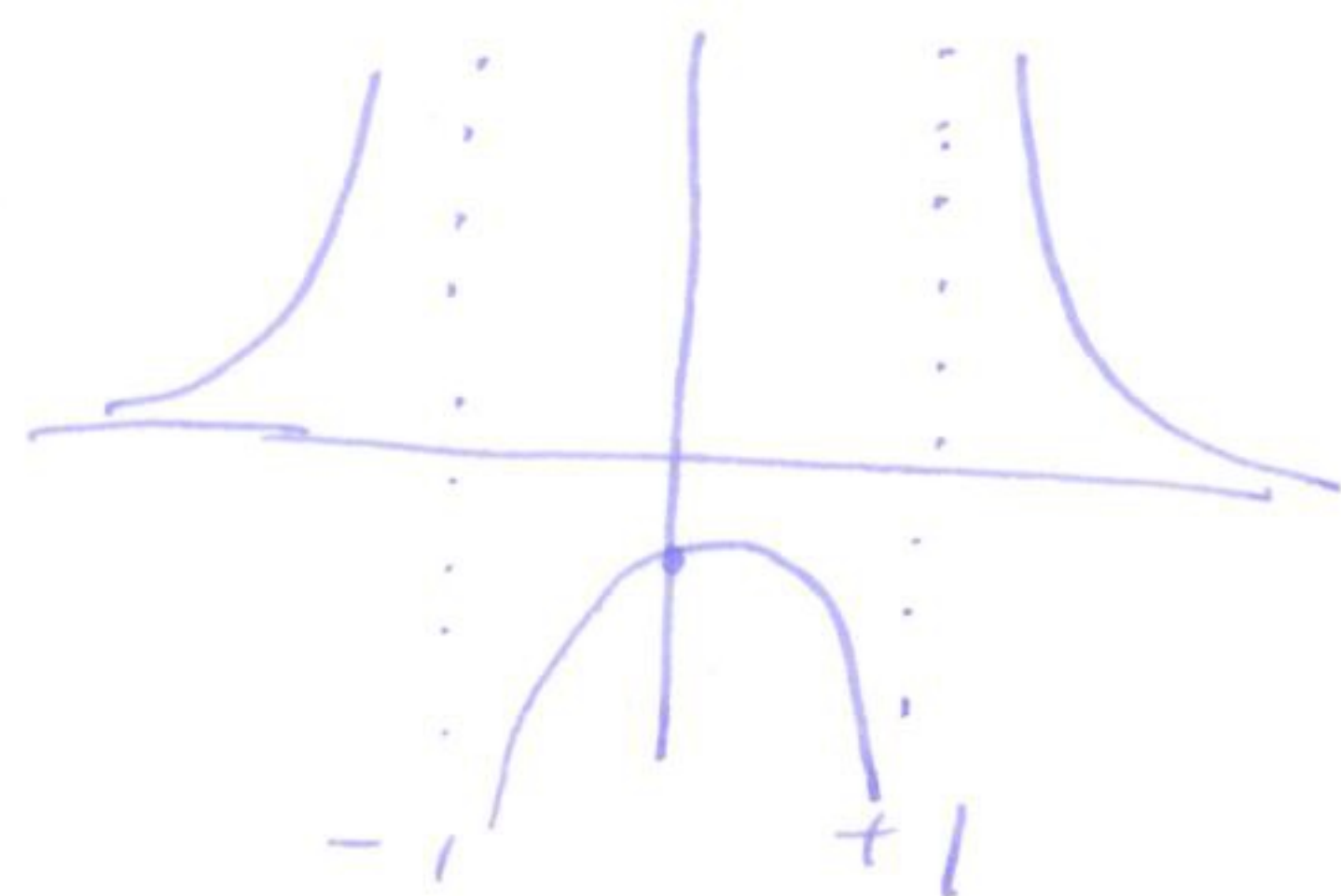
$$f'(x) = - (x^2-1)^{-2} \cdot 2x = \frac{-2x}{(x^2-1)^2}$$

critical point at $x = 0$

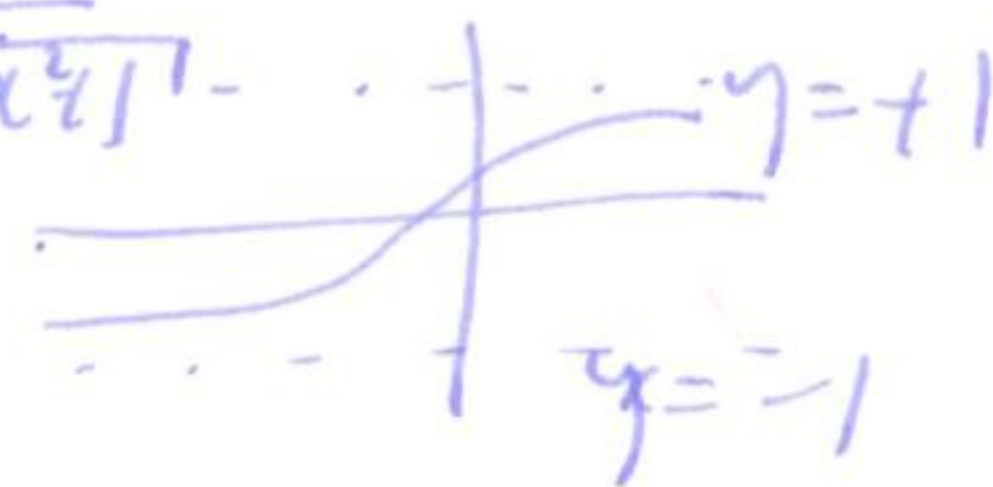
first derivative test: $\begin{array}{c} f'(x) \\ \hline x=0 \end{array}$ local max. $f(0) = -1$

$$f''(x) = \frac{(x^2-1)^2 \cdot (-2) - (-2x) \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4} = \frac{6x^2+2}{(x^2-1)^3}$$

sign $\begin{array}{c} + \quad | \quad - \quad | \quad + \\ -1 \quad \quad +1 \end{array}$



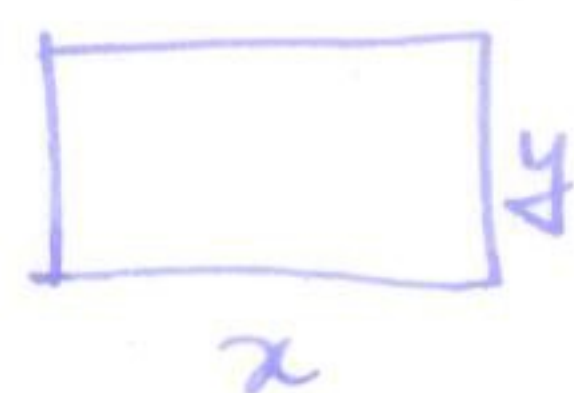
Example $f(x) = \frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{1+x^2}}$



§4.6 Optimization

Example A piece of wire of length L is bent into a rectangle.

What's the largest area rectangle we can make?



$$2x + 2y = L$$

$$\text{area} = xy$$

i.e. max $A = xy$ such that $2x + 2y = L$

$$\Rightarrow y = \frac{L}{2} - x$$

$$\text{So max } A = x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2$$

find max: $A'(x) = \frac{L}{2} - 2x$ given $0 \leq x \leq \frac{L}{2}$ (check endpoints)

$$\text{solve } A'(x) = 0 \Rightarrow x = \frac{L}{4} \Rightarrow y = \frac{L}{4} \text{ (square)}$$

Example what shape of cylindrical tin minimizes surface area, if you want total volume to be 1 ft^3 ?



$$V = \pi r^2 h = 1 \Rightarrow h = \frac{1}{\pi r^2}$$

$$A = 2\pi r h + 2\pi r^2$$

So $A = 2\pi r \left(\frac{1}{\pi r^2} \right) + 2\pi r^2 = \frac{2}{r} + 2\pi r^2$

$$A'(r) = -\frac{2}{r^2} + 4\pi r$$

solve $A'(r) = 0$

$$-\frac{2}{r^2} + 4\pi r = 0$$

$$4\pi r = \frac{2}{r^2}$$

$$r^3 = \frac{1}{2\pi} \quad r = \sqrt[3]{\frac{1}{2\pi}}$$

§4.7 L'Hôpital's rule

Thm Suppose $f(x)$ and $g(x)$ are differentiable and

$f(a) = g(a) = 0$, and $g'(x) \neq 0$ for all x near a , then
(or $f(a) = g(a) = \pm\infty$) $\leftarrow$ [i.e. indeterminate form]

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \quad (\text{provided this limit exists})$$

Warning: this is not the quotient rule! We are not differentiating

$$\frac{f(x)}{g(x)}$$

Examples ① $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{3x^2}{1} = 2$