§4.5 Graph sketching and asymptotes

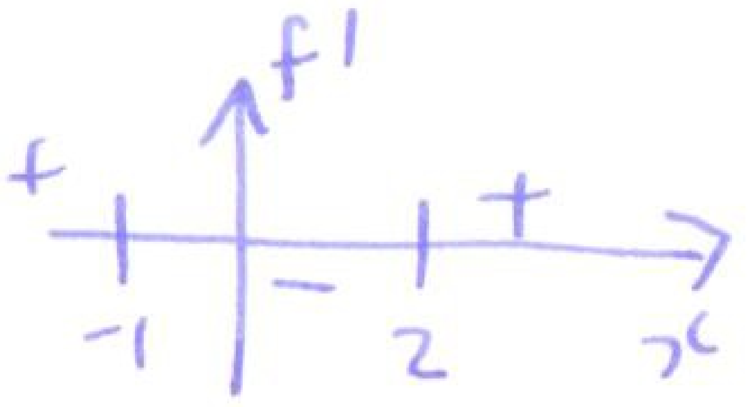
Example sketch graph of $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$

(i.e. find critical points and signs of f', f'')

$f'(x) = x^2 - x - 2 = (x+1)(x-2)$ critical points at $x = -1, 2$

$f''(x) = 2x - 1$ $\quad f''(-1) = -3 < 0 \Rightarrow x = -1$ is local max

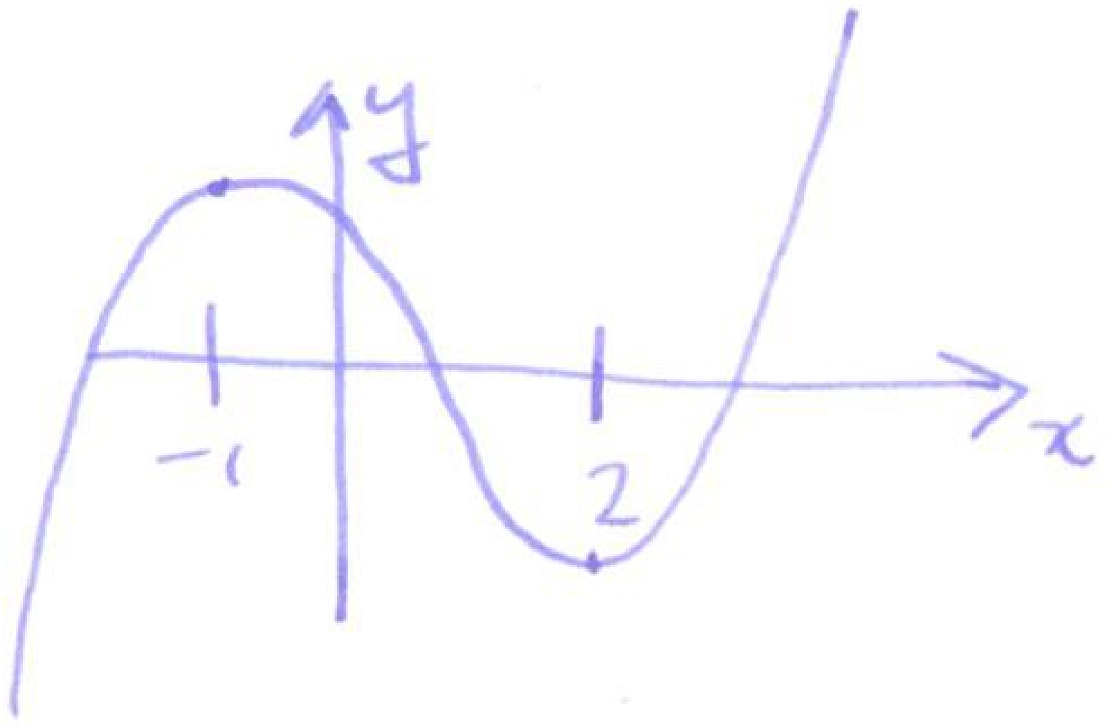
$f''(2) = 3 > 0 \Rightarrow x = 2$ is local min



$f(-1) = -\frac{1}{3} - \frac{1}{2} + 2 + 3 > 0$

$f(2) = \frac{8}{3} - 2 - 4 + 3 < 0$

$x^x = e^{x\ln(x)}$



critical point example

$f(x) = (4x - x^2)e^x$

$f'(x) = (4 + 2x - x^2)e^x.$

$x = 1 \pm \sqrt{5}.$

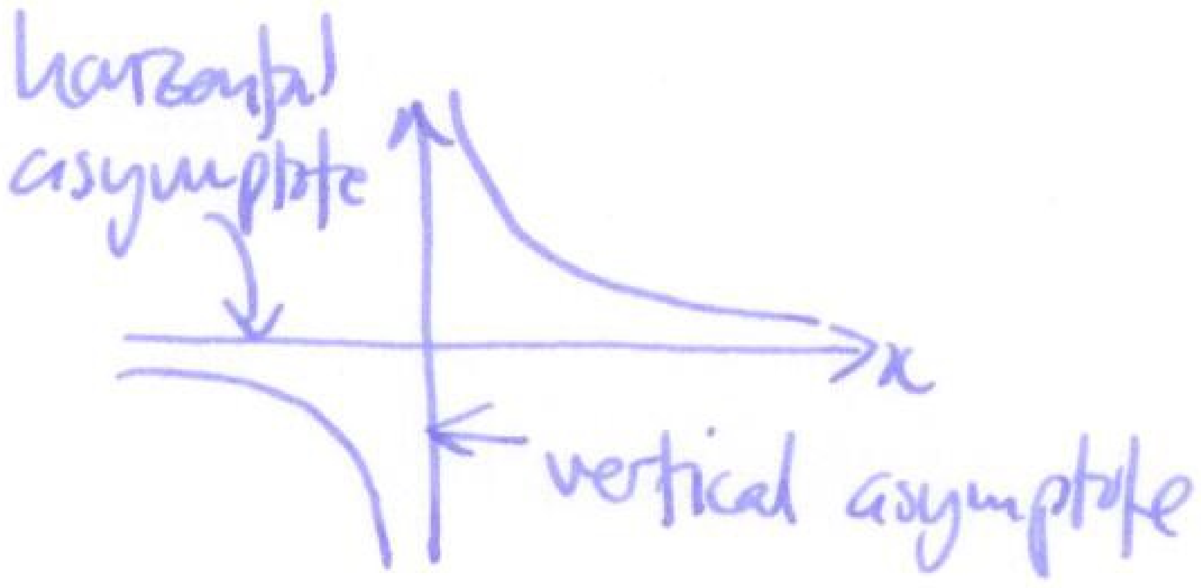
$f''(x) = (6 - x^2)e^x$

$f''(1+\sqrt{5}) < 0$ local max

$f''(1-\sqrt{5}) > 0$ local min

Asymptotes

Example $y = \frac{1}{x}$



$y = c$ is a **horizontal asymptote** if $\lim_{x \to \infty} f(x) = c$

or $\lim_{x \to -\infty} f(x) = c$

$x = c$ is a **vertical asymptote** if $\lim_{x \to c+} f(x) = \pm\infty$

or $\lim_{x \to c-} f(x) = \pm\infty$.

Observation rational functions $\frac{p(x)}{q(x)}$ can have horizontal asymptotes only if $\deg(p) \leq \deg(q)$

Example $\frac{x^2+x+1}{3x^2+2} = \frac{1+1/x+1/x^2}{3+2/x^2} \to \frac{1}{3}$ as $x \to \pm\infty$.

$\frac{x^3+x}{x^2-2} = \frac{x+1/x}{1-2/x^2} \to +\infty$ as $x \to \infty$
$-\infty$ $x \to -\infty$.

Example sketch graph of $\frac{3x+2}{2x-4}$

① find vertical asymptotes (i.e. solve denominator = 0)

$2x-4=0 \Rightarrow x=2$

② find $f'(x) = \frac{(2x-4).3-(3x+2).2}{(2x-4)^2} = \frac{-16}{(2x-4)^2} = \frac{-4}{(x-2)^2}$

$f'(x)=0$ no solutions so no critical points

$f'(x)$ is always -ve.

find $f''(x) = -4.-2(x-2)^{-3} = \frac{8}{(x-2)^3}$ $x>2$ +ve concave up
$x<2$ -ve down.

③ horizontal asymptotes: $\lim_{x\to\pm\infty} \frac{3+2/x}{2-4/x} = \frac{3}{2}$.