hyperbolic trig functions

$y = \sinh(x) = \dfrac{e^x - e^{-x}}{2}$

$\dfrac{dy}{dx} = \cosh(x)$

$y = \cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$\dfrac{dy}{dx} = \sinh(x)$

$y = \sinh^{-1}(x)$

$\sinh(y) = x$ $\quad [\cosh^2(y) - \sinh^2(y) = 1]$

$\cosh(y)\dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\cosh(y)} = \dfrac{1}{\sqrt{1+x^2}}$

$y = \cosh^{-1}(x)$

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2-1}}$, similarly

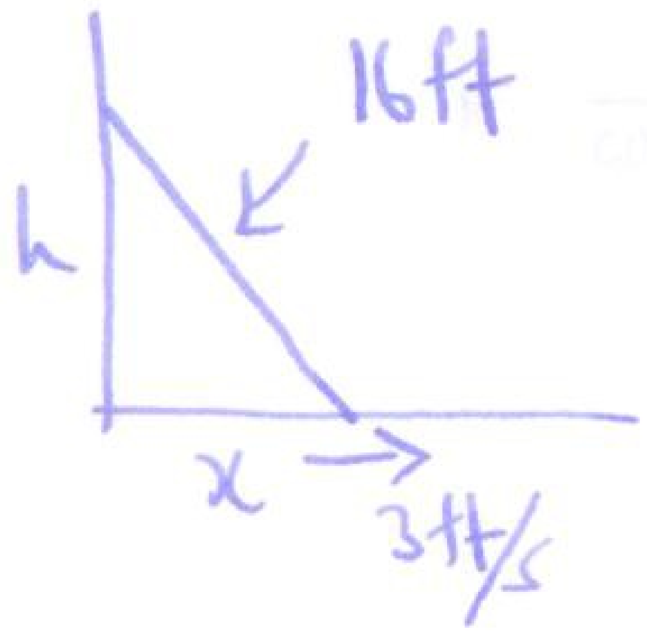
$y = \tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

$\dfrac{dy}{dx} = \dfrac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)}$

$\dfrac{dy}{dx} = \dfrac{1}{\cosh^2(x)} = \operatorname{sech}^2(x)$

§3.11 Related rates

Example falling ladders.



$x(t)$ distance of foot of ladder from wall

$h(t)$ height of top of ladder

suppose $x = 5$, $\dfrac{dx}{dt} = 3$ ft/s

Q: what is $\dfrac{dh}{dt}$?

pythagoras: $x^2 + h^2 = 16^2$

differentiate wrt t: $2x\dfrac{dx}{dt} + 2h\dfrac{dh}{dt} = 0$

plug in: $x = 5$

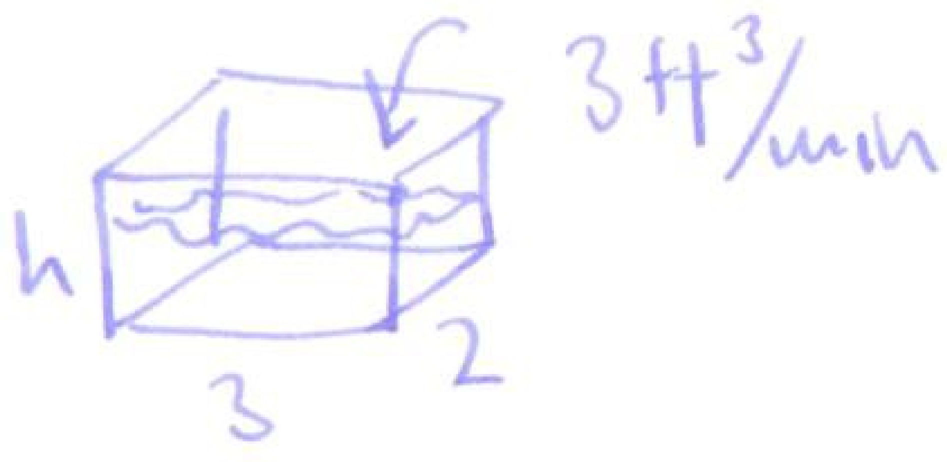
$\dfrac{dx}{dt} = 3$

$h = \sqrt{16^2 - 5^2} \simeq 15.2$

$2 \cdot 5 \cdot 3 + 2 \overset{15.2}{h} \dfrac{dh}{dt} = 0 \Rightarrow \dfrac{dh}{dt} = -\dfrac{30}{30.4} \simeq -1$ ft/s

related rates: · need equation that relates variables
· implicit differentiation
· plug in info

Example: filling a rectangular box



Q: how fast is the water rising?

V = volume of water $V(t)$

$\frac{dV}{dt}$ = rate of change of water w/time = 3 ft/min

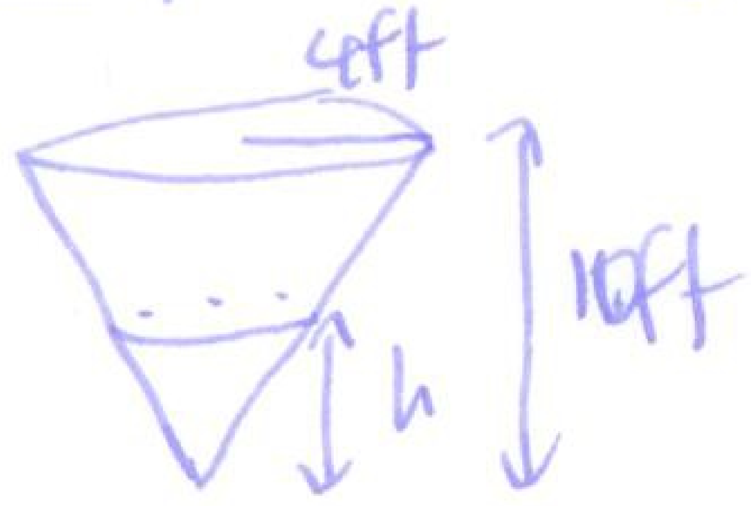
h = height of water

$\frac{dh}{dt}$ = rate of change of height w/time.

$V = 3 \cdot 2 \cdot h$ i.e. $V = 6h$

so $\frac{dV}{dt} = 6\frac{dh}{dt}$ $3 = 6\frac{dh}{dt}$ $\frac{dh}{dt} = \frac{1}{2}$ ft/min

Example: filling a conical tank: Q how fast is the water rising when $h=5$?



water in at 10 ft³/min i.e. $\frac{dV}{dt} = 10\,\text{ft}^3/\text{min}$

volume of cone: $V = \frac{1}{3}\pi h r^2$

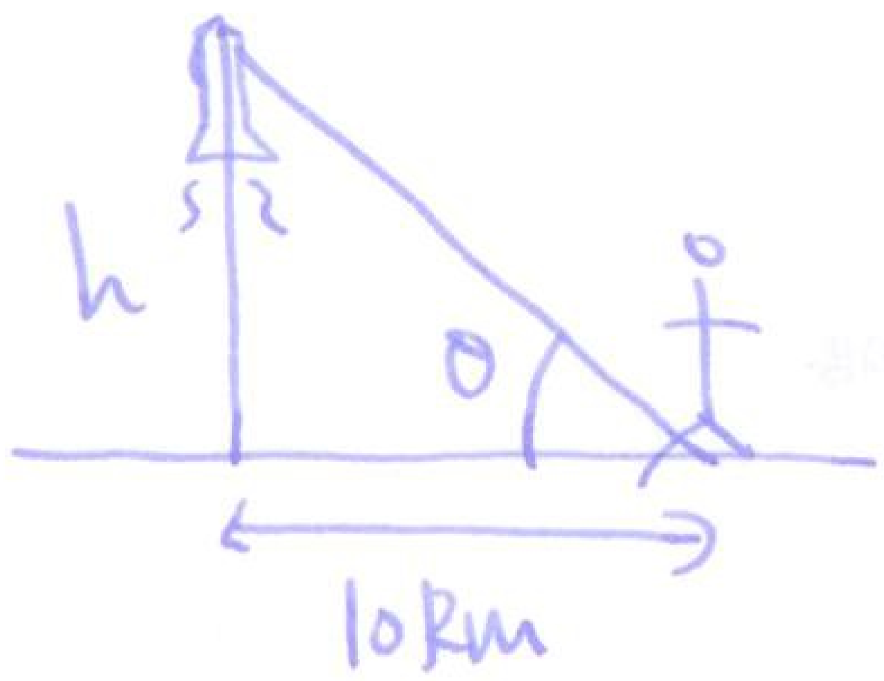
need: r in terms of h $\frac{r}{h} = \frac{4}{10} = \frac{2}{5}$ (similar triangles)

$$V = \frac{1}{3}\pi h\left(\frac{2h}{5}\right)^2 = \frac{4\pi h^3}{75}$$

$$\frac{dV}{dt} = \frac{4\pi}{75}\cdot 3h^2\frac{dh}{dt} = \frac{4\pi h^2}{25}\frac{dh}{dt}$$

so when $h=5$
water rising at rate $\frac{10}{4\pi}$ ft/min

Example tracking rockets

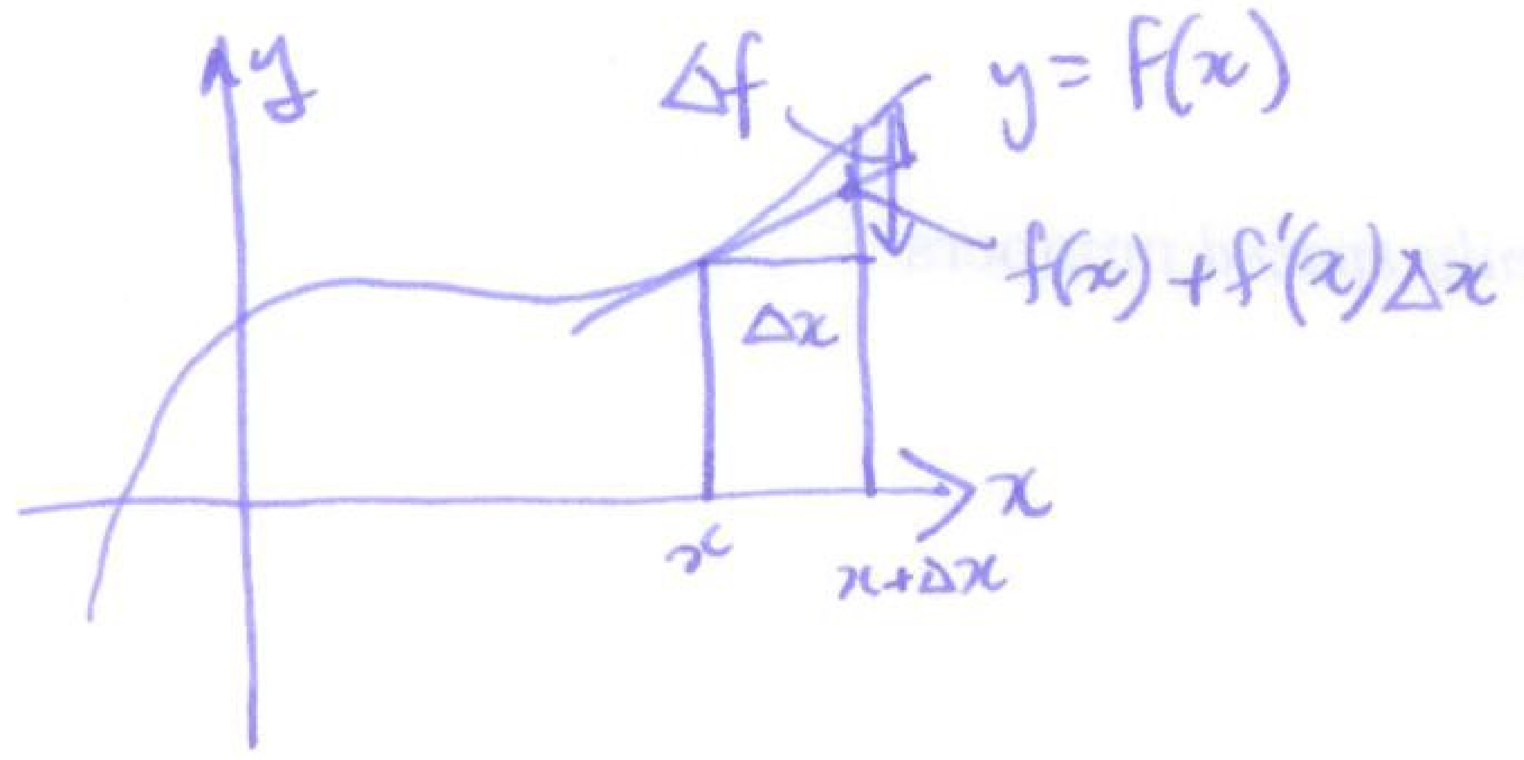


if angle is $\frac{\pi}{3}$ and rate of change of angle $\frac{1}{2}$ radians/min, how fast is the rocket going?

$$\frac{h}{10} = \tan\theta \qquad \frac{1}{10}\frac{dh}{dt} = \sec^2\theta \, \frac{d\theta}{dt}$$

$$\frac{dh}{dt} = 10 \sec^2\left(\frac{\pi}{3}\right) \cdot \frac{1}{2} \approx 20 \text{km/min}$$

§4.1 Linear approximations



If f is differentiable at x, and Δx is small then $\boxed{\Delta f \approx f'(x)\,\Delta x}$

Example what is $\sqrt{103}$?

$$f(x) = \sqrt{x} = x^{1/2} \qquad f(100) = \sqrt{100} = 10$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad f'(100) = \frac{1}{20}$$

so $\sqrt{103} = f(100+3) \approx f(100) + f'(100) \cdot 3$

$$\approx 10 + \frac{1}{20} \cdot 3 = 10.15$$