Last time: movement under gravity

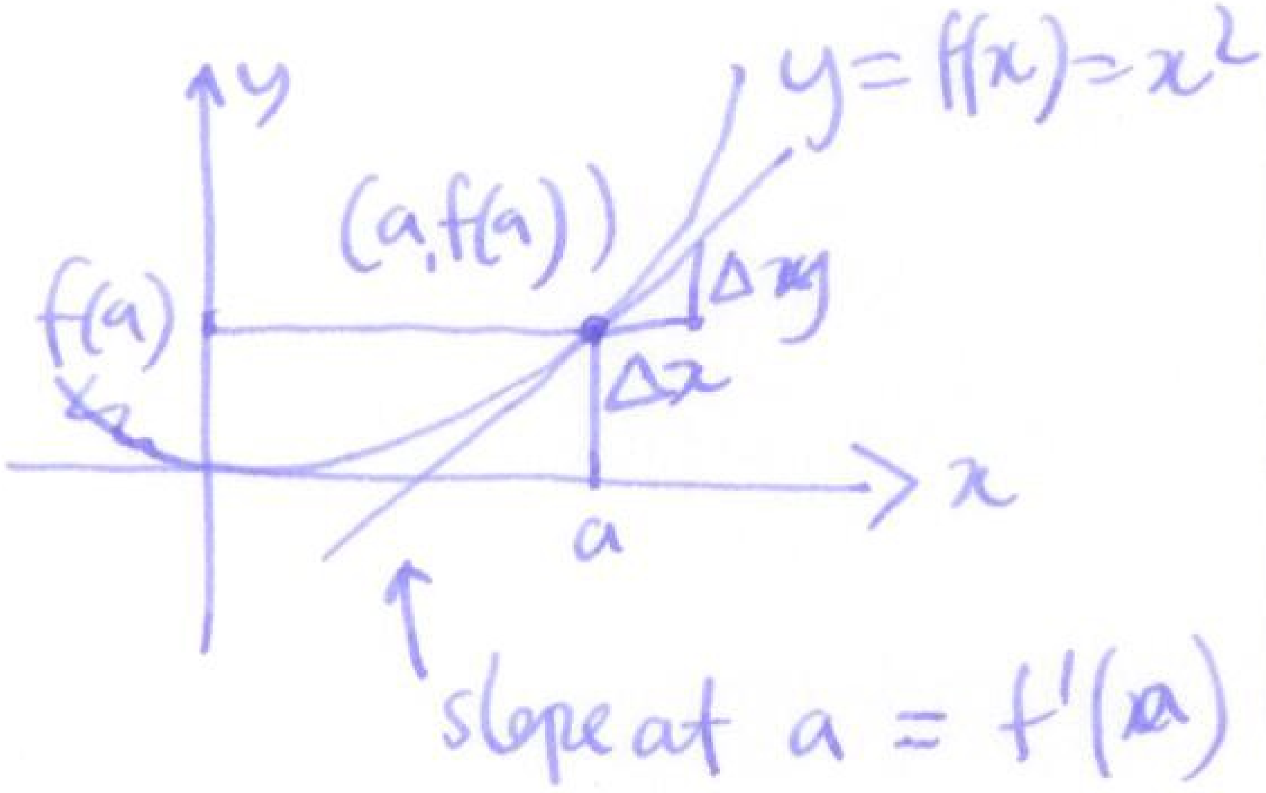
$$h''(t) = -g \qquad (g = 9.8\,m/s,\ 32\,ft/s)$$
$$h'(t) = -tg + v_0 \qquad v_0 = \text{initial velocity at } t=0$$
$$h(t) = -\tfrac{1}{2}tg^2 + v_0 t + h_0 \qquad h_0 = \text{initial height at } t=0$$

§ 3.9 Derivatives of inverse functions

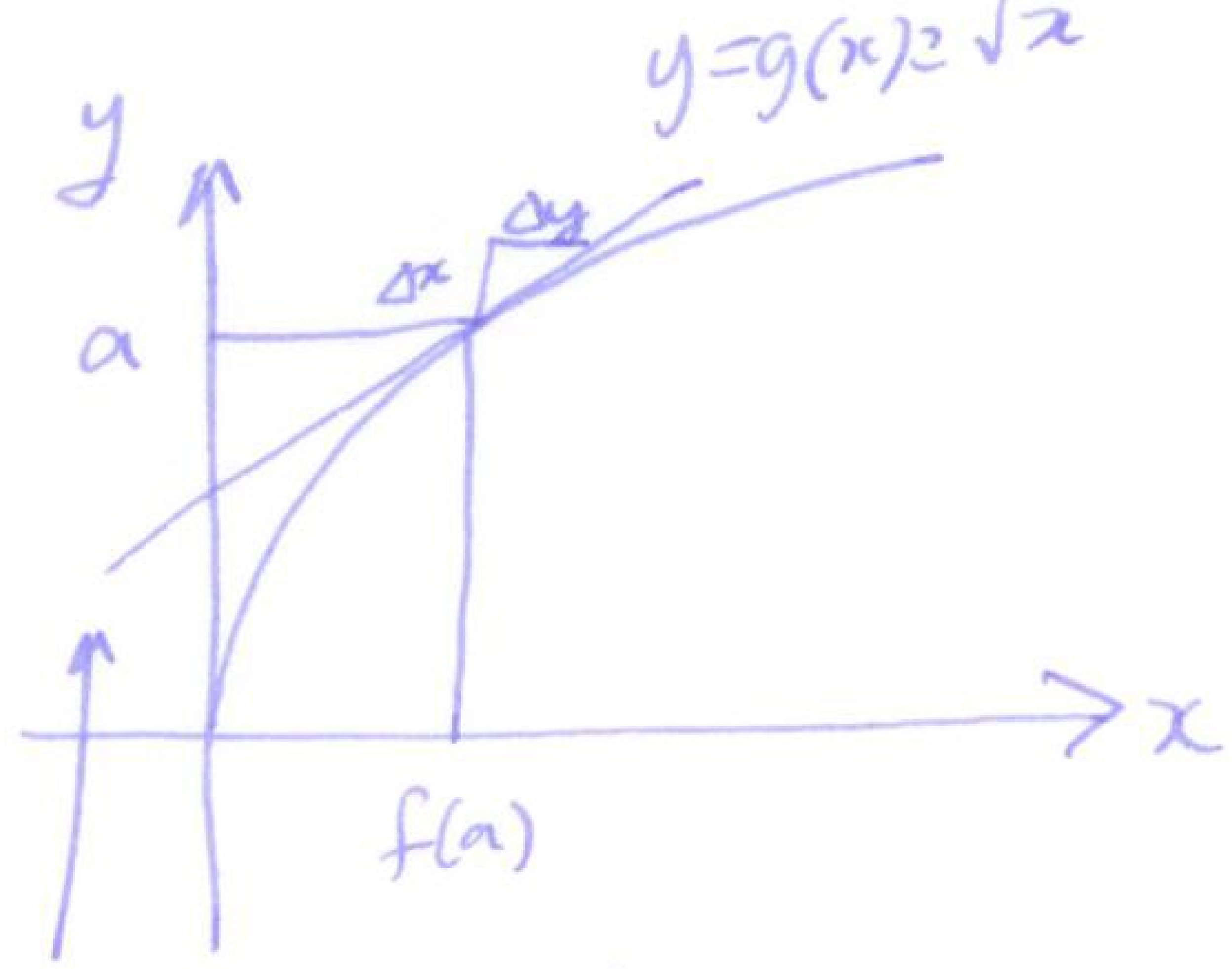
Thm $f(x)$ differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$

then $g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{f'(f^{-1}(x))}$ as long as $f'(g(x)) \neq 0$

why?



inverse
⇝
reflect in $y = x$



slope at $f(a) = \dfrac{1}{f'(a)}$
$\shortparallel$
$g'(f(a)) = \dfrac{1}{f'(a)}$

rewrite

$$g'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Example

$f(x) = x^2$ (on $[0,\infty)$) $\quad f'(x) = 2x$

$g(x) = f^{-1}(x) = \sqrt{x}$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{2g(x)} = \frac{1}{2\sqrt{x}}$$

<u>Proof</u> $y = f(x)$

$g(y) = f^{-1}(y) = x \;\}$ use implicit differentiation

$$g'(y)\frac{dy}{dx} = 1$$

$$\cancel{\frac{dy}{dx} = \frac{1}{g'(y)}} \quad \left[\frac{dy}{dx} = f'(x)\right]$$

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$$

rewrite:

$$g'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(g(x))}$$

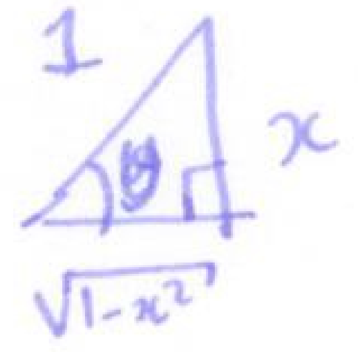
<u>Thm</u> $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$

<u>Proof</u> $y = \sin^{-1}(x)$

$$\sin(y) = x$$

$$\cos(y)\frac{dy}{dx} = 1$$

$\frac{dy}{dx} = \frac{1}{\cos(y)}$ use: $\cos^2(y) + \sin^2(y) = 1$ or:



$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \square$$

Thm: $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{x^2+1}$ $\qquad$ $\frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{x^2+1}$

$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}}$ $\qquad$ $\frac{d}{dx}\left(\csc^{-1}(x)\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

Proof $\quad y = \tan^{-1}(x)$

$$\tan(y) = x$$

$$\sec^2(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} \qquad \text{use: } \sin^2 y + \cos^2 y = 1$$
$$\tan^2 y + 1 = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \qquad \square$$

Example $\frac{d}{dx}\left(\tan^{-1}\left(e^{2x}\right)\right)$: use · chain rule

· $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$

$$= \frac{1}{1+(e^{2x})^2}\cdot\left(e^{2x}\right)' = \frac{2e^{2x}}{1+e^{4x}}$$

§3.10 Derivatives of exponentials and logs

recall: · if $f(x) = b^x$ then $f'(x) = m_b\, b^x$

· $\frac{d}{dx}\left(e^x\right) = e^x$

Thm: $\frac{d}{dx}\left(b^x\right) = \ln(b)\, b^x$

Proof $b^x = e^{\ln(b)x}$

use chain rule: $\frac{d}{dx}\left(b^x\right) = e^{\ln(b)x} \cdot \ln(b)$

$= \ln(b)\, b^x$. $\square$.

Example $f(x) = 2^x = e^{\ln(2)x}$

$f'(x) = \ln(2) e^{\ln(2)x} = \ln(2)\, 2^x$.

Thm $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$

Proof $y = \ln(x)$

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ $\square$

Example $\frac{d}{dx}\left(x\ln(x)\right) = 1 + \ln(x)$