so $h'(x) = f'(g(x)) \cdot g'(x)$

$f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-1/2}$

$g(x) = x^3 + 1$

$g'(x) = 3x^2$

so

$$h'(x) = \frac{1}{2}\left(g(x)\right)^{-1/2} \cdot 3x^2$$

$$= \frac{1}{2} \frac{1}{\sqrt{x^3+1}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3+1}}$$

Alternate notation $h(x) = f(g(x))$ set $u = g(x)$

then $f(g(x)) = f(u)$, where $u = g(x)$

$$h'(x) = f'(g(x)) \cdot g'(x) = f'(u)\, u'(x)$$

$$\frac{dh}{dx} = \cancel{f'(u(x))} \cdot \frac{df}{du} \cdot \frac{du}{dx}$$

mnemonic : "match num/denom of fractions".

Examples ① $\frac{d}{dx}(\cos(x^2)) = -\sin(x^2) \cdot 2x$

② $\frac{d}{dx}\left(e^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{x^{1/2}}\right) = e^{x^{1/2}} \cdot \frac{1}{2} x^{-1/2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

③ $\frac{d}{dx}\left(\sqrt{x+\sqrt{x^2+1}}\right) = \frac{d}{dx}\left(\left(x+(x^2+1)^{1/2}\right)^{1/2}\right)$.

$$= \frac{1}{2}\left(x+(x^2+1)^{1/2}\right)^{-1/2} \cdot \frac{d}{dx}\left(x + (x^2+1)^{1/2}\right).$$

$$= \frac{1}{2}\left(x+(x^2+1)^{1/2}\right)^{-1/2} \cdot \left(1 + \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x\right).$$

Proof (of chain rule)

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

[answer should be $f'(g(x)) . g'(x)$]

rewrite this as : $$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \underbrace{\frac{g(x+h)-g(x)}{h}}_{\lim_{h\to 0} (*) = g'(x)}$$

set $k = g(x+h) - g(x)$

then as g is continuous $h \to 0 \Rightarrow k \to 0$

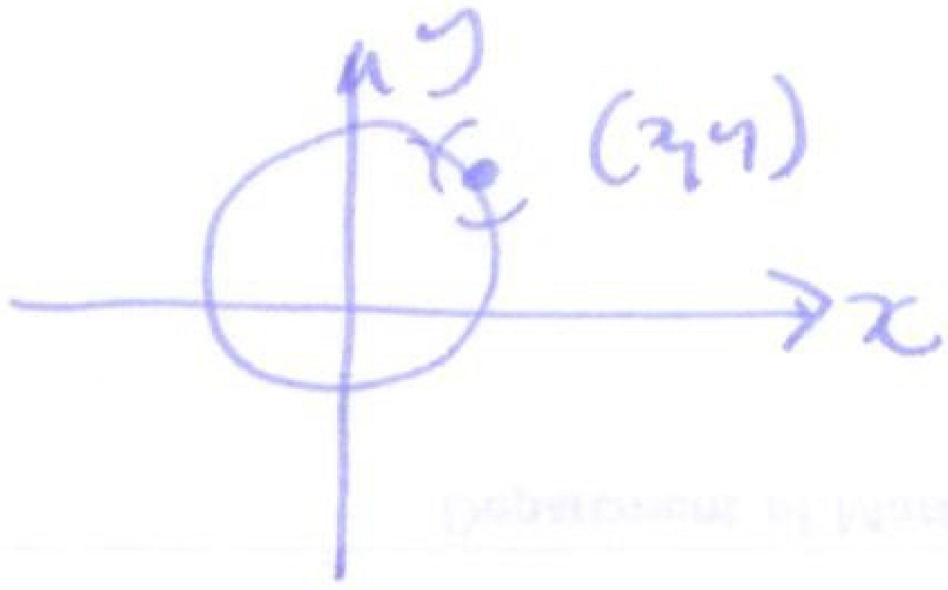
then $$\lim_{k\to 0} \frac{f(g(x)+k) - f(g(x))}{k} = f'(g(x))$$ $\square$

More examples $$\frac{d}{dx}\left((g(x))^n \right) = n\,(g(x))^{n-1} . g'(x)$$

$$\frac{d}{dx}\left(e^{g(x)} \right) = e^{g(x)} . g'(x)$$

$$\frac{d}{dx}\left(f(kx+b) \right) = k f'(kx+b)$$

§3.8 Implicit differentiation



explicit function : $f(x) = x^2 + x + 1$

implicit "function" : $x^2 + y^2 = 1$

defines "local" functions near a point (x,y) on the curve we can think of the y-values as a function of x, i.e. $(x, y(x))$.

differentiate: $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$

$$2x + \frac{d}{dx}\left(y(x)^2\right) = 0$$

$$2x + 2y(x)\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

still implicit!

<u>Example</u> Find tangent line to circle $x^2+y^2=1$

at the point $\left(\frac{3}{5}, \frac{4}{5}\right)$ $\quad \frac{dy}{dx} = -\frac{x}{y} = \frac{-3/5}{4/5} = -3/4$

so $\quad y - \frac{4}{5} = -\frac{3}{4}\left(x - \frac{3}{5}\right)$

<u>Example</u> find tangent line to $y^4 + xy = x^3 - x + 2$ at $(1,1)$

implicit differentiation: $\quad 4y^3y' + y + xy' = 3x^2 - 1$

set $x=1, y=1$: $\quad 4y' + 1 + y' = 3 - 1$

$$5y' = 1 \qquad y' = \frac{1}{5}$$

tangent line: $\quad y - 1 = \frac{1}{5}(x-1)$