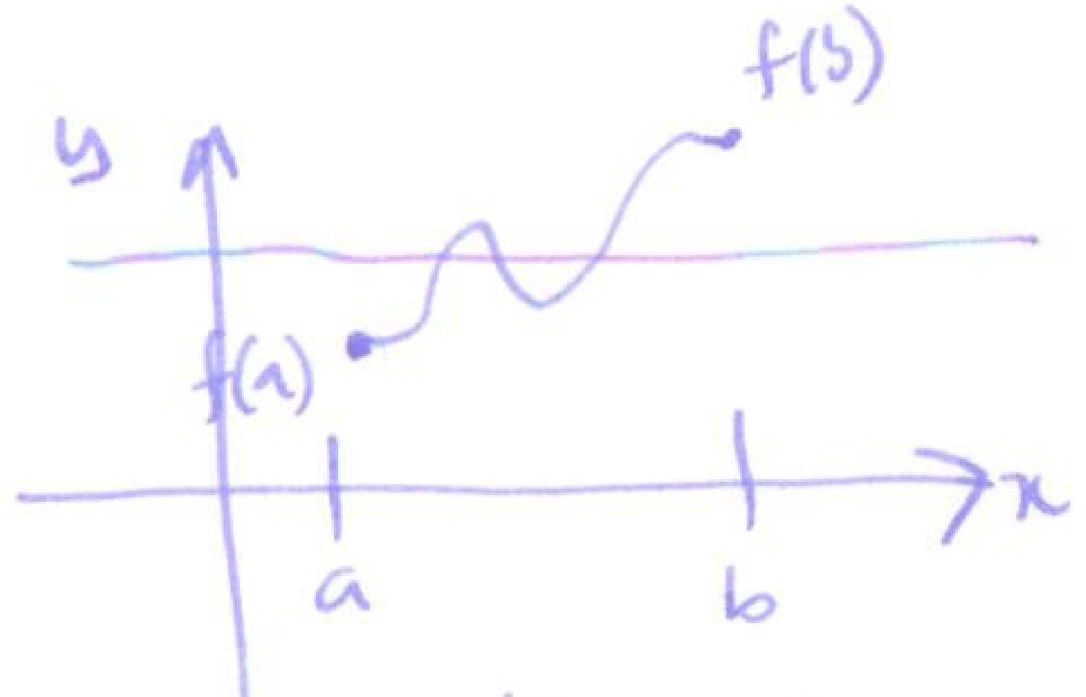
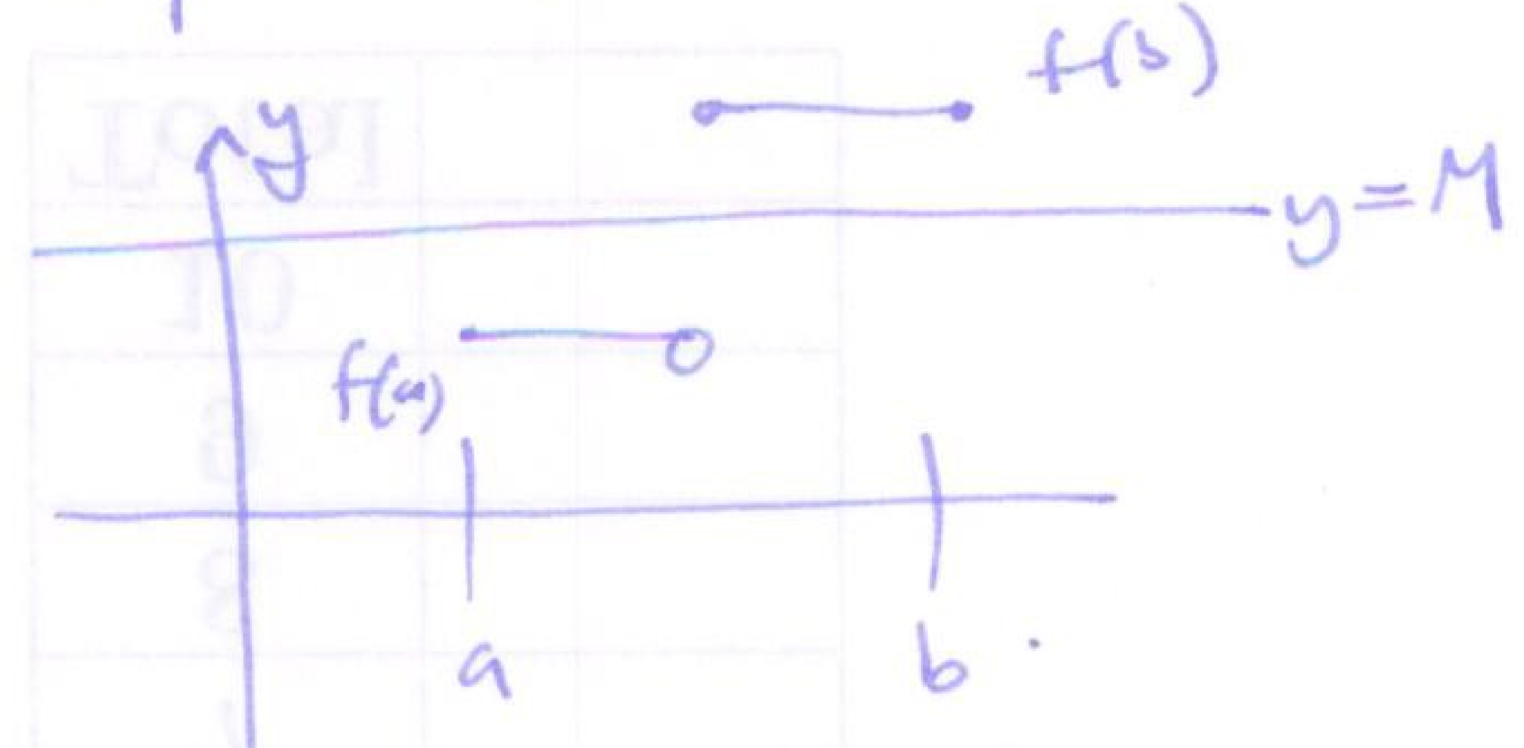
§2.7 Intermediate value theorem

"continuous functions can't skip values".



continuous



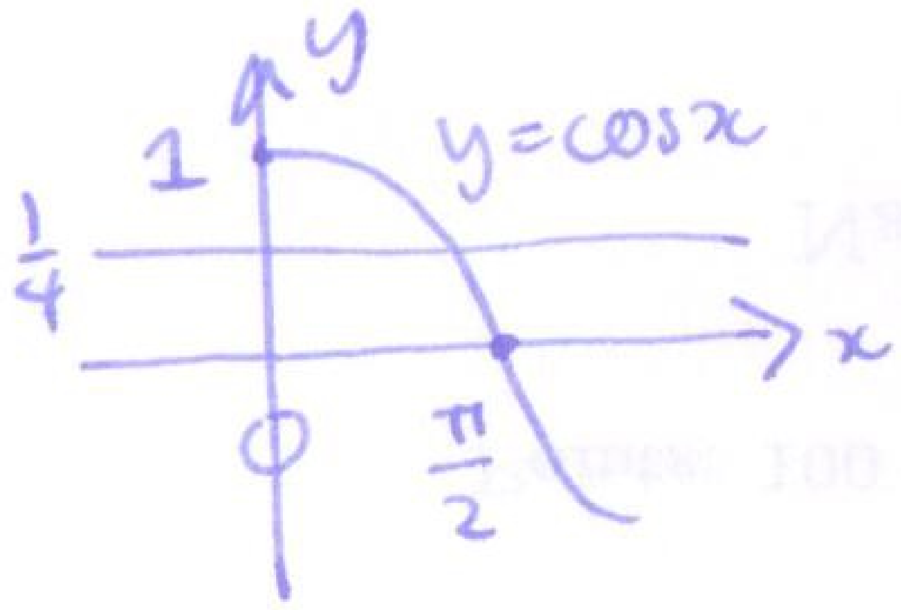
not continuous.

<u>Thm</u> (Intermediate value theorem IVT)

If $f(x)$ is a continuous function on a closed interval $[a,b]$ and $f(a) \neq f(b)$, then for any M between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ such that $f(c) = M$.

<u>Application</u> : showing equations have solutions.

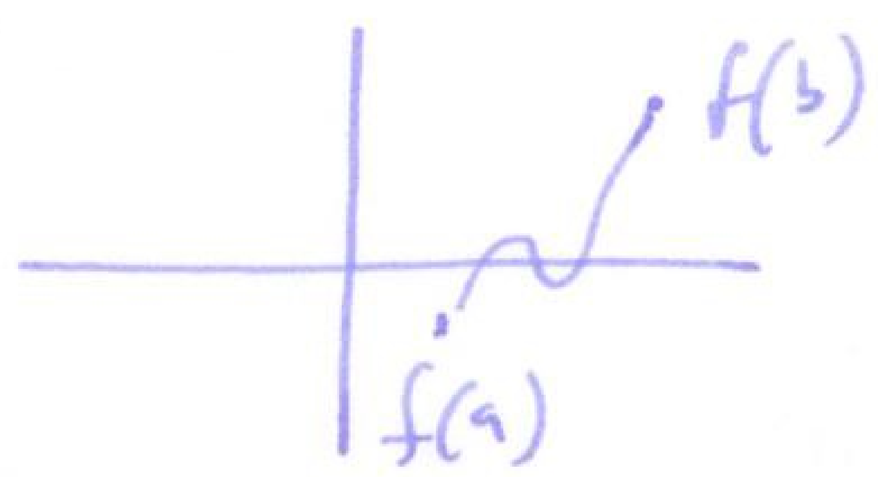
<u>Example</u> show that $\cos(x) = 1/4$ has at least one solution in $[0, \frac{\pi}{2}]$



$$\left.\begin{array}{l} \cos(0) = 1 \\ \cos(\frac{\pi}{2}) = 0 \\ \cos x \text{ cts} \end{array}\right\} \Rightarrow \cos(x) = \frac{1}{4} \text{ has at least one sol}^{\underline{n}} \text{ in } [0, \tfrac{\pi}{2}] \text{ by IVT.}$$

<u>special case</u> : finding zeros.

<u>Corollary</u> If $f(x)$ is cts, and $f(a), f(b)$ have different signs, then there is at least one $c \in [a,b]$ s.t. $f(c) = 0$.



<u>Application</u> : bisection method

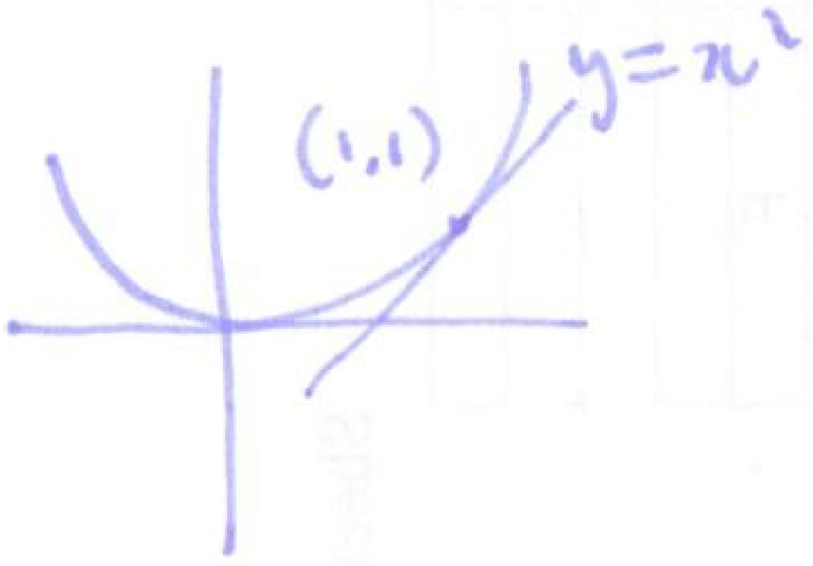
look for a sol$^{\underline{n}}$ to $\sin x = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

<u>notation</u> slope of tangent line at $x=a$ written $f'(a)$ (newton) or $\frac{df}{dx}(a)$ [Liebnitz]

if this limit exists, we say f is <u>differentiable</u> at $x=a$.
if the limit does not exist, then f is <u>not</u> differentiable.

<u>note</u>: $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$ equivalent to $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$

<u>Example</u> find slope of tangent line to $y=x^2$ at $x=1$.



$$\lim_{\substack{h \to 0 \\ x=1}} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h}$$

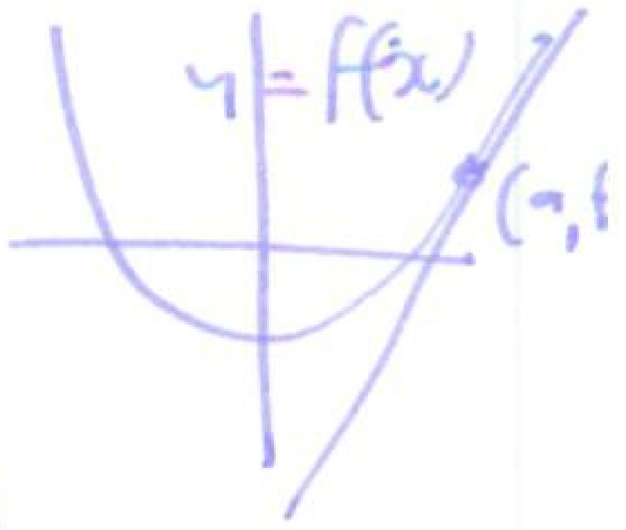
$$= \lim_{h \to 0} \frac{1+2h+h^2-1}{h}$$

$$= \lim_{h \to 0} 2+h = 2.$$

<u>Def$^{\underline{n}}$</u> the tangent line to $f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$

The equation of this line is $y - y_0 = m(x - x_0)$.

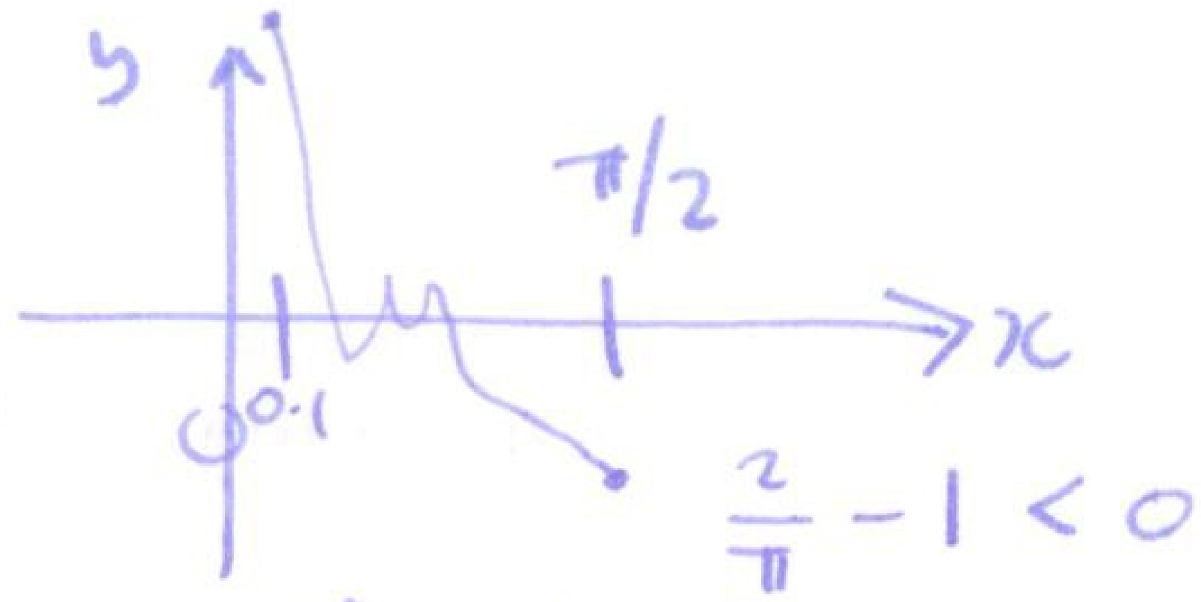
$$y - f(a) = f'(a)(x-a)$$



<u>Example</u> find tangent line to (1,1) for $y=x^2$. slope $= 2$

$$y - 1 = 2(x-1)$$

consider $\frac{1}{x} - \sin x = 0$
$f(x)$ "



so $f(0.1)$ and $f(\frac{\pi}{2})$ have different signs

compute $f\left(\frac{0.1+\frac{\pi}{2}}{2}\right) = f\left(\frac{\pi}{4}+0.05\right)$ +ve choose $\left[\frac{\pi}{4}+0.05, \frac{\pi}{2}\right]$

−ve choose $\left[0.1, \frac{\pi}{4}+0.05\right]$

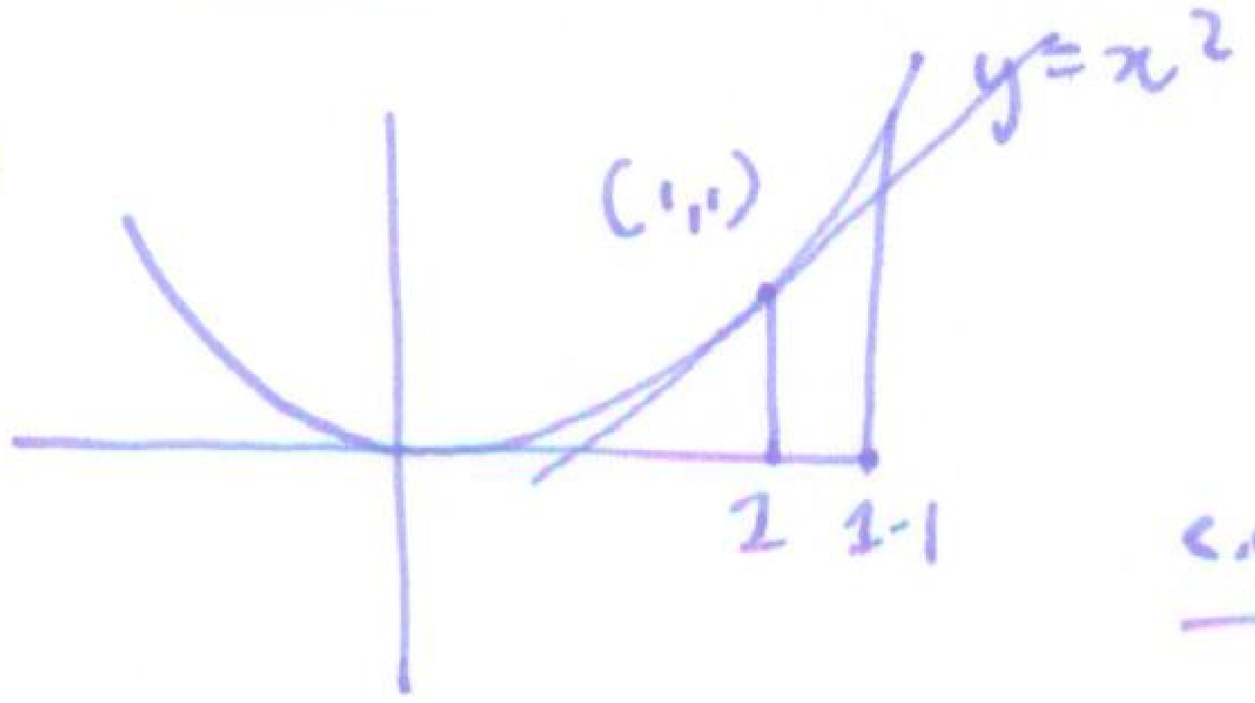
and so on...

Example show $\sqrt{x} + \sqrt{x+1} = 2$ for some number x.

x	$f(x)$
0	1
1	$1+\sqrt{2} \sim 2.4$.

§3.1 Definition of the derivative

recall :



we can compute average rate of change of a function over some interval $[x_1, x_2]$

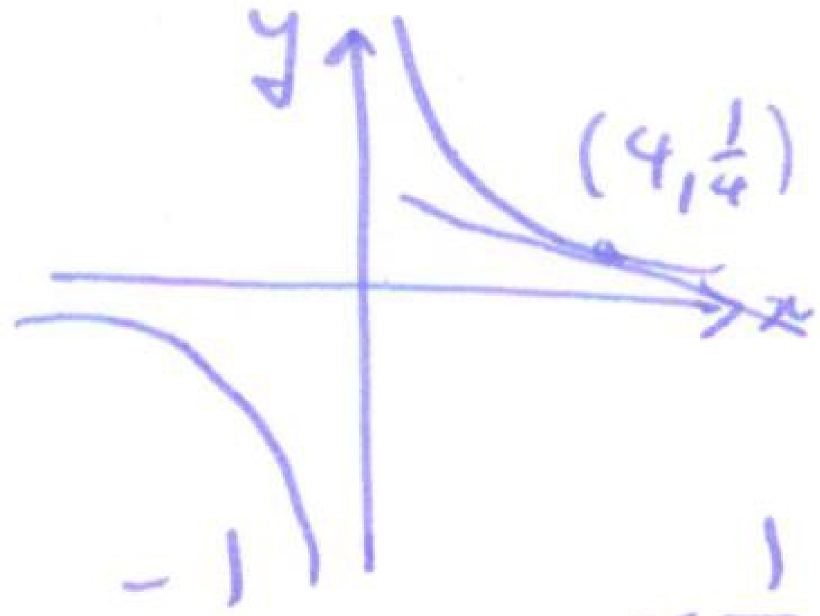
e.g. $\frac{f(1.1) - f(1)}{1.1 - 1} = \frac{(1.1)^2 - 1}{0.1} \approx 2$

Q: how do we compute the slope of the tangent line at $(x, f(x))$?

idea : look at average rate of change on a small interval $[x, x+h]$ and take limit as $h \to 0$.

Def^{n}: slope of tangent line at $x = a$ is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Example find slope of tangent line to $y = 1/x$ at $(4, 1/4)$

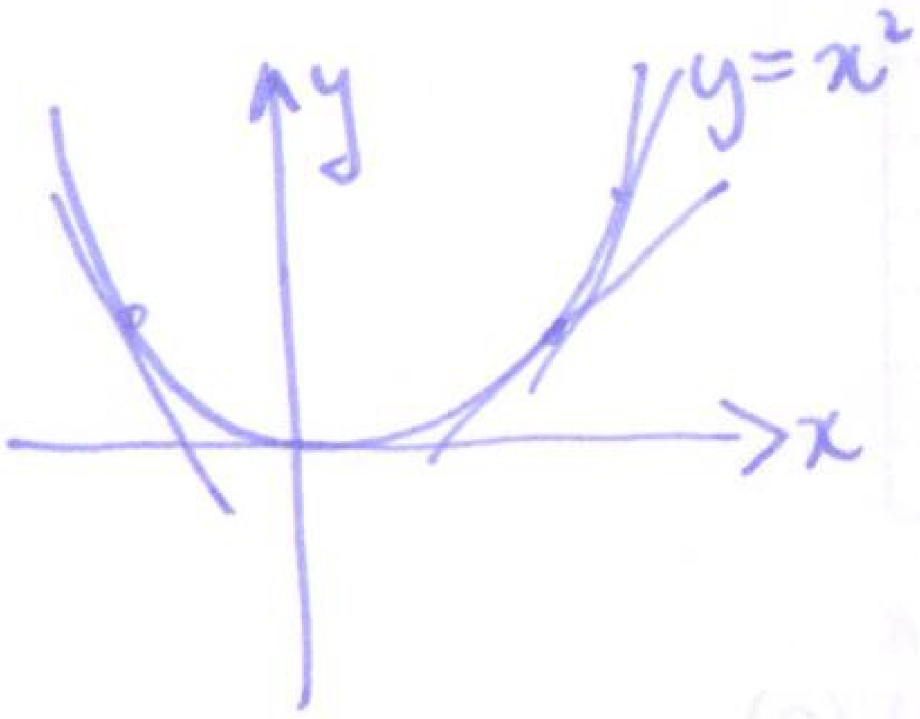


find $f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h}$

$= \lim_{h\to 0} \frac{1}{h} \frac{4-(4+h)}{4(4+h)} = \lim_{h\to 0} \frac{-h}{h\,4(4+h)} = \lim_{h\to 0} \frac{-1}{16+4h} = -\frac{1}{16}$

tangent line : $y - y_0 = m(x-x)$ $\qquad y - \frac{1}{4} = -\frac{1}{16}(x-4)$

§ 3.2 Derivative as a function



$f : \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$

at each point x, there is the slope of the tangent line at $(x, f(x))$ so we can define

a function $f' : \mathbb{R} \to \mathbb{R}$
$x \mapsto$ slope of tangent line at $(x, f(x))$

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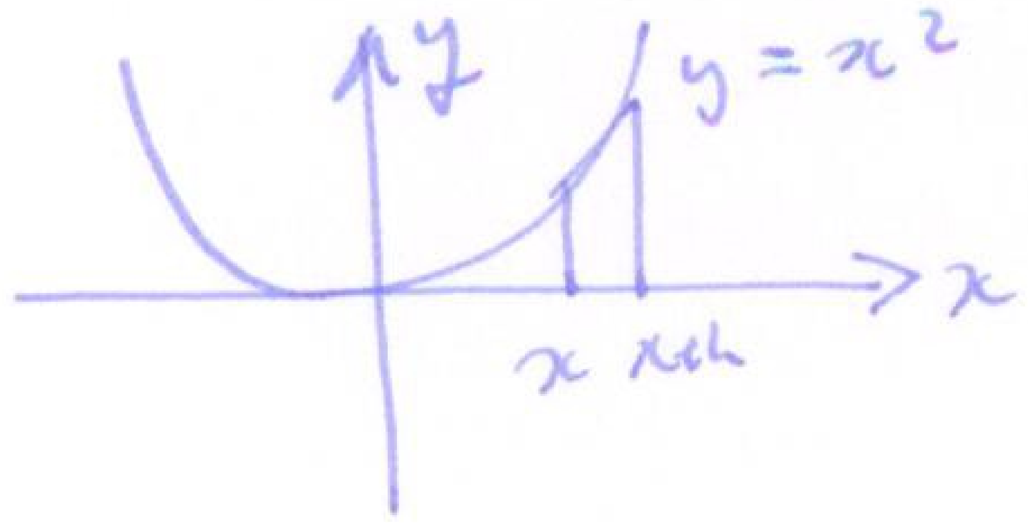
notation : $f'(x)$ say " derivative of f "

Example

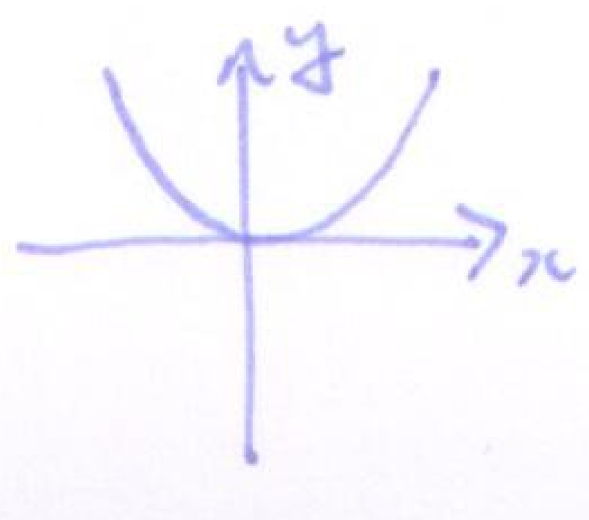


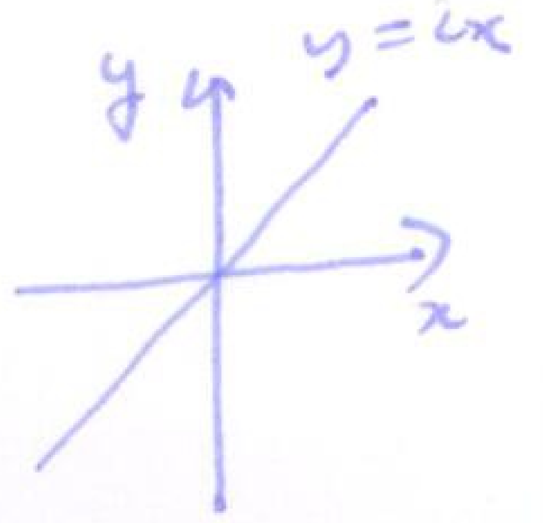
slope at x.

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

if $f(x) = x^2$: $f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim_{h\to 0} 2x + h = 2x$





so if $f(x) = x^2$ then $f'(x) = 2x$.