<u>Warning</u>: none of this works if either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ does not exist. (or is infinite)

<u>Bad example</u> $\quad 1 = x \cdot \frac{1}{x} \quad (x \neq 0) \qquad \lim_{x\to 0} 1 = 1$

$\lim_{x\to 0} x = 0 \qquad \lim_{x\to 0} \frac{1}{x}$ DNE.

<u>Observation</u> . sum and product rules hold for multiple functions

$$\lim_{x\to c}\left(f_1(x)+f_2(x)+f_3(x)\right) = \lim_{x\to c} f_1(x) + \lim_{x\to c} f_2(x) + \lim_{x\to c} f_3(x)$$

etc.

. difference: $\lim_{x\to c}\left(f(x)-g(x)\right) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

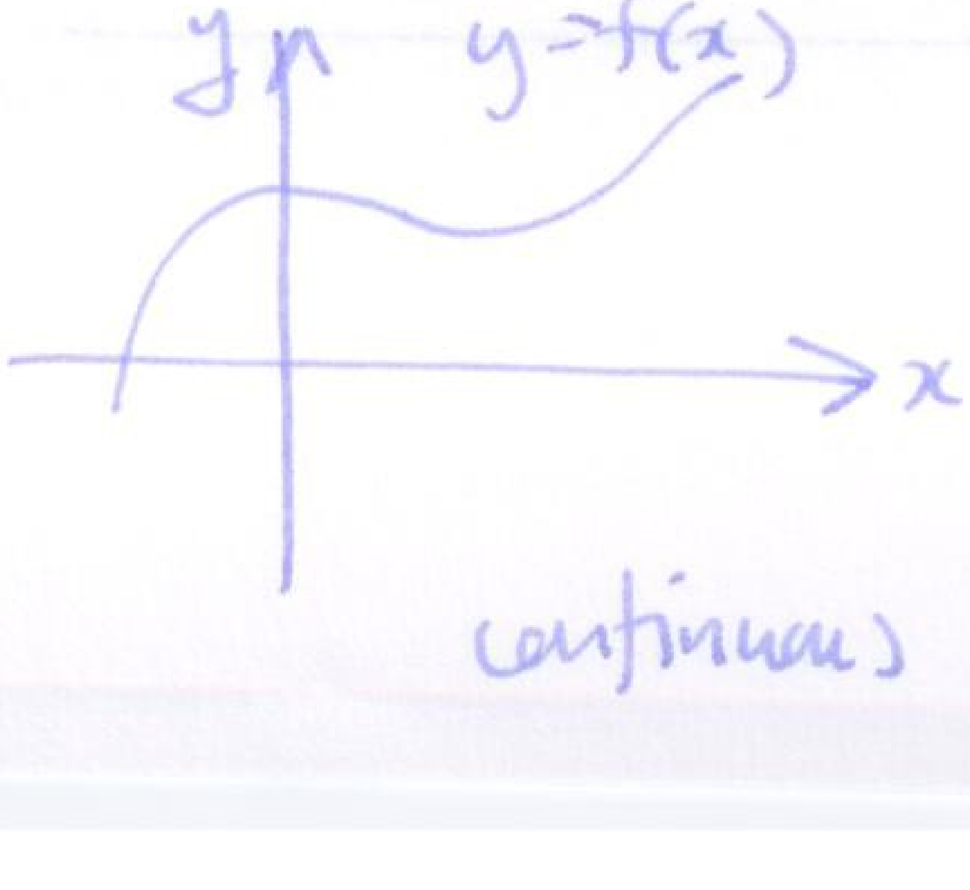
(use sum and constant multiple $k=-1$)

<u>Example</u> $\quad \lim_{x\to 3} x^2 = \lim_{x\to 3} x \cdot \lim_{x\to 3} x = 3.3 = 9.$

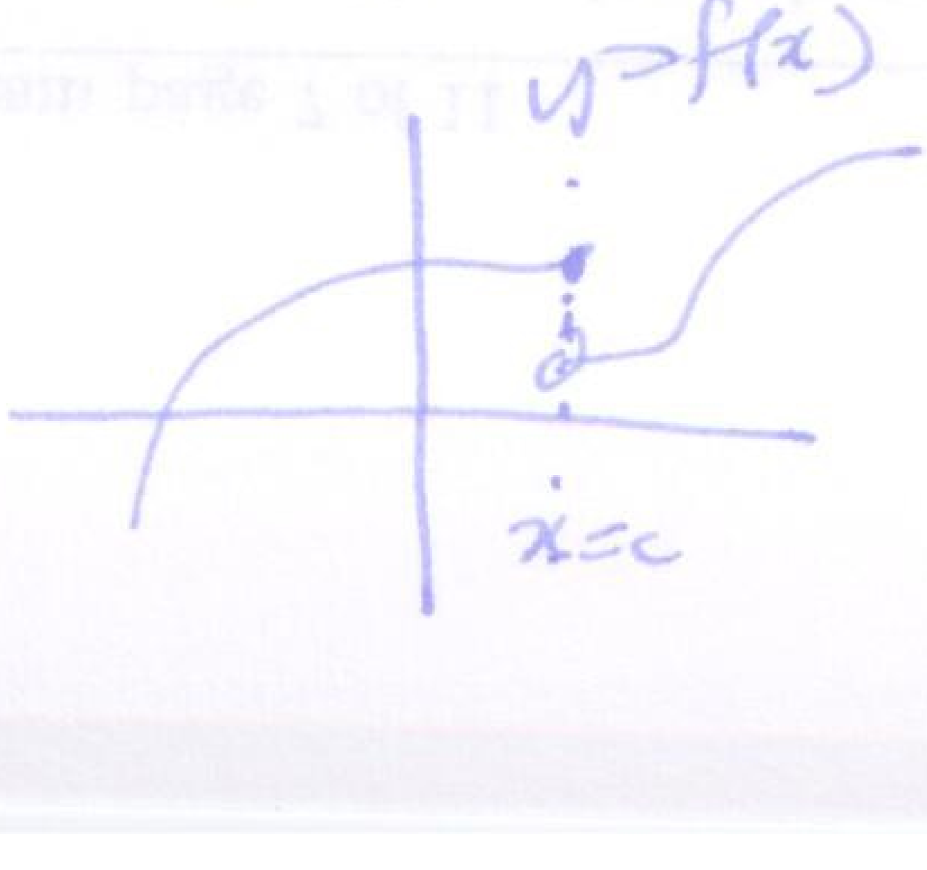
$$\lim_{t\to 2} \frac{t+5}{3t} = \frac{\lim_{t\to 2} t+5}{\lim_{t\to 2} 3t} = \frac{7}{6}$$

<u>§2.4 Limits and continuity</u>

<u>Example</u>



continuous



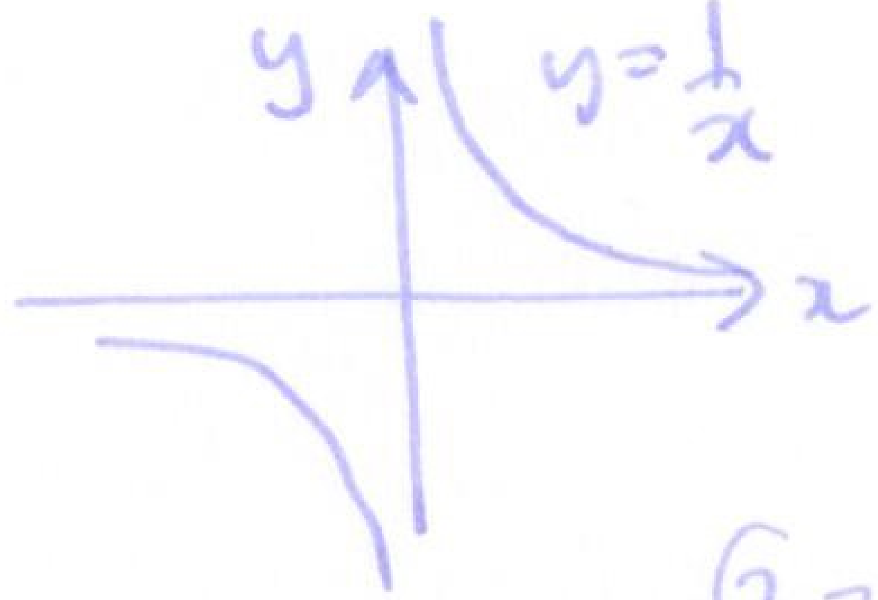
<u>not</u> continuous at $x=c$.

Defⁿ we say $f(x)$ is cts at $x=c$ if

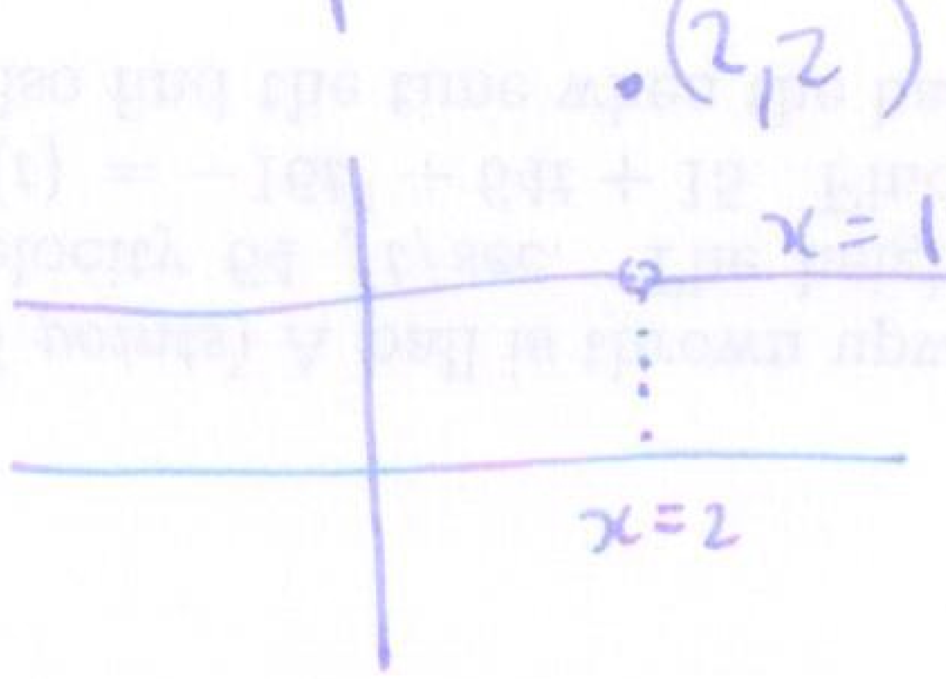
$$\lim_{x\to c} f(x) = f(c)$$

If the limit does not exist, or is not equal to $f(c)$, then f is <u>not</u> continuous at c (discontinuous at c)

Example



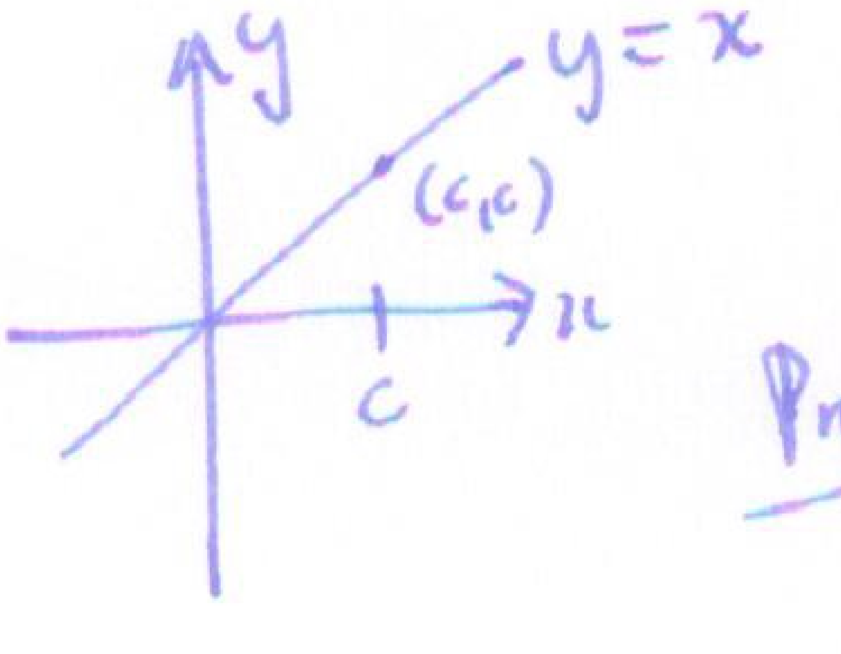
<u>not</u> cts at $x=0$



$$f(x) = \begin{cases} 1 & (x \neq 2) \\ 2 & x = 2 \end{cases}$$

<u>not</u> cts at $x=2$

Example show $f(x) = x$ is cts



$f(c) = c$ so want to show $\lim_{x\to c} f(x) = c$

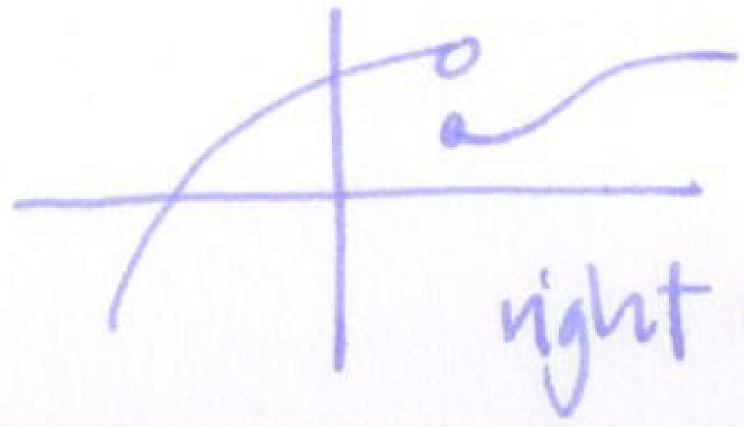
Proof: need $|f(x) - L|$ to get small as $x \to c$

$|x - c| \to 0$ as $x \to c$ ✓.

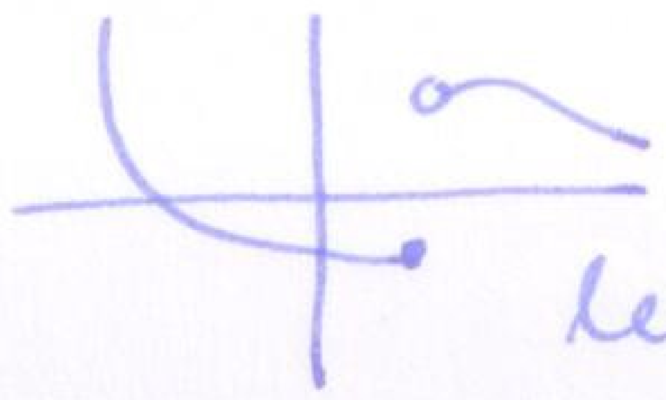
Defⁿ $f(x)$ is left continuous at $x=c$ if $\lim_{x\to c-} f(x) = f(c)$

right continuous $\lim_{x\to c+} f(x) = f(c)$

Example



right cts



left cts.

Building continuous functions

Thm 1 Suppose that $f(x)$ and $g(x)$ are both cts at $x=c$

Then the following functions are cts at c

1) $f(x)+g(x)$
2) $kf(x)$ for any ~~bean~~ constant k
3) $f(x)g(x)$
4) $f(x)/g(x)$ if $g(\cancel{x}c)\neq 0$

Proof these follow directly from the limit laws.

e.g. if $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ both exist then

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \quad . \quad \text{etc.} \ \square .$$

Thm 2 Polynomials $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ are cts.

Rational functions $p(x)/q(x)$ are cts where $q(x)\neq 0$.

Proof $f(x)=x$ is cts.

so $f(x)f(x) = x^2$ is cts $\Rightarrow$ $x^3, x^4, \dots$ cts.

so sums and constant multiples are cts... $\square$.

similarly $p(x)/q(x)$ cts where $q(x)\neq 0$ $\square$.

Useful facts

Thm 3 (Basic functions)

- $\sin(x)$, $\cos(x)$ are cts
- b^x cts $(b>0)$
- $\log_b(x)$ cts $(b>0)$
- $x^{1/n}$ cts.

Thm 4 (Inverse functions)

If $f: D \to R$ is continuous with ~~inf~~ inverse $f^{-1}: R \to D$ then f^{-1} is cts.

Thm 5 (Composition) $f(g(x))$

If g is continuous at $x=c$ and f is continuous at $x=g(c)$ then $f(g(x))$ is continuous at $x=c$.

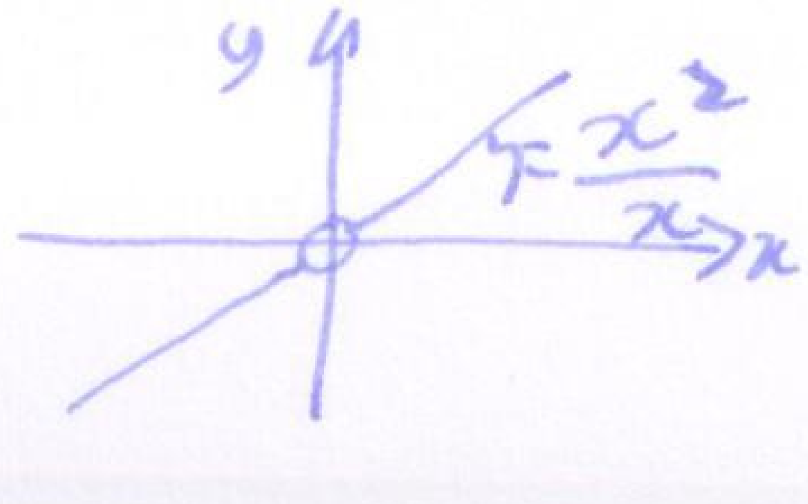
— // —

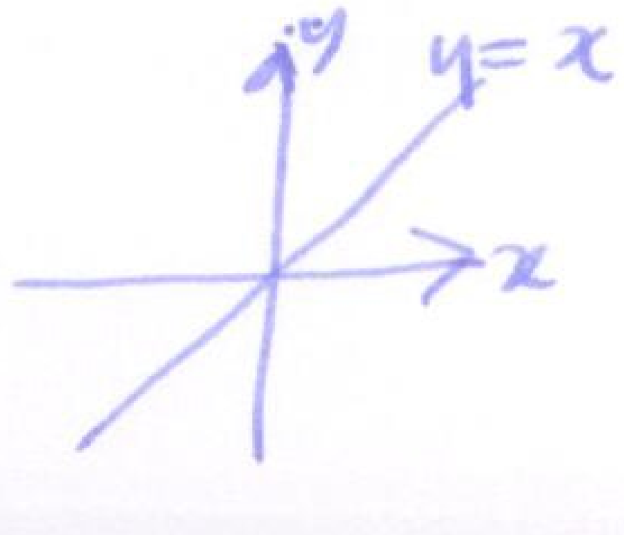
Limits of continuous functions are easy to calculate: just substitute.

Example $f(x) = \dfrac{2^x + \sin(x)}{\sqrt{x^2+x+1}}$ $\qquad \displaystyle\lim_{x\to 1} f(x) = \frac{2+\sin(1)}{\sqrt{3}}$

§2.5 Evaluating limits algebraically

Example $\displaystyle\lim_{x\to 0} \frac{x^2}{x}$ $\qquad$ note: $\dfrac{x^2}{x} = \underbrace{\dfrac{0}{0}}_{\text{indeterminate form}}$ undefined when $x=0$





but limit does not depend on value at 0

so $\dfrac{x^2}{x} = \dfrac{x}{1} = x$ for $x \neq 0$

so $\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$

other indeterminate forms : $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty\cdot 0$, $\infty-\infty$.

Example : $\lim_{x\to 3} \frac{x^2-4x+3}{x^2+x-12}$ $\qquad \frac{9-12+3}{9+3-12} = \frac{0}{0}$ indeterminate.

factor : $\frac{(x-3)(x-1)}{(x-3)(x+4)} = \frac{x-1}{x+4} \quad (x\neq 3)$ so $\lim_{x\to 3}\frac{x-1}{x+4} = \frac{2}{7}$

Example : $\lim_{x\to \frac{\pi}{2}} \frac{\tan x}{\sec x}$ $\qquad \tan(\pi/2) = \frac{\sin(\pi/2)}{\cos(\pi/2)} = 1/0$

$\sec(\pi/2) = \frac{1}{\cos(\pi/2)} = 1/0.$

$\frac{\infty}{\infty}$ indet form.

note : $\frac{\tan(x)}{\sec(x)} = \frac{\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos x}} = \sin(x) \quad (\text{if } \cos(x)\neq 0)$

so $\lim_{x\to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \sin x = 1.$

Example : $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}$ $\qquad \lim_{x\to 5} \frac{x-5}{\sqrt{x+4}-3}\cdot\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = 6$

$= \frac{1}{4}$

Example $\lim_{x\to 1} \frac{1}{x-1} - \frac{2}{x^2-1} = \frac{x+1-2}{x^2-1} = \frac{x-1}{x^2-1} = \frac{1}{x+1} = \frac{1}{2}.$

$(\infty-\infty)$