§8.3 Complex numbers

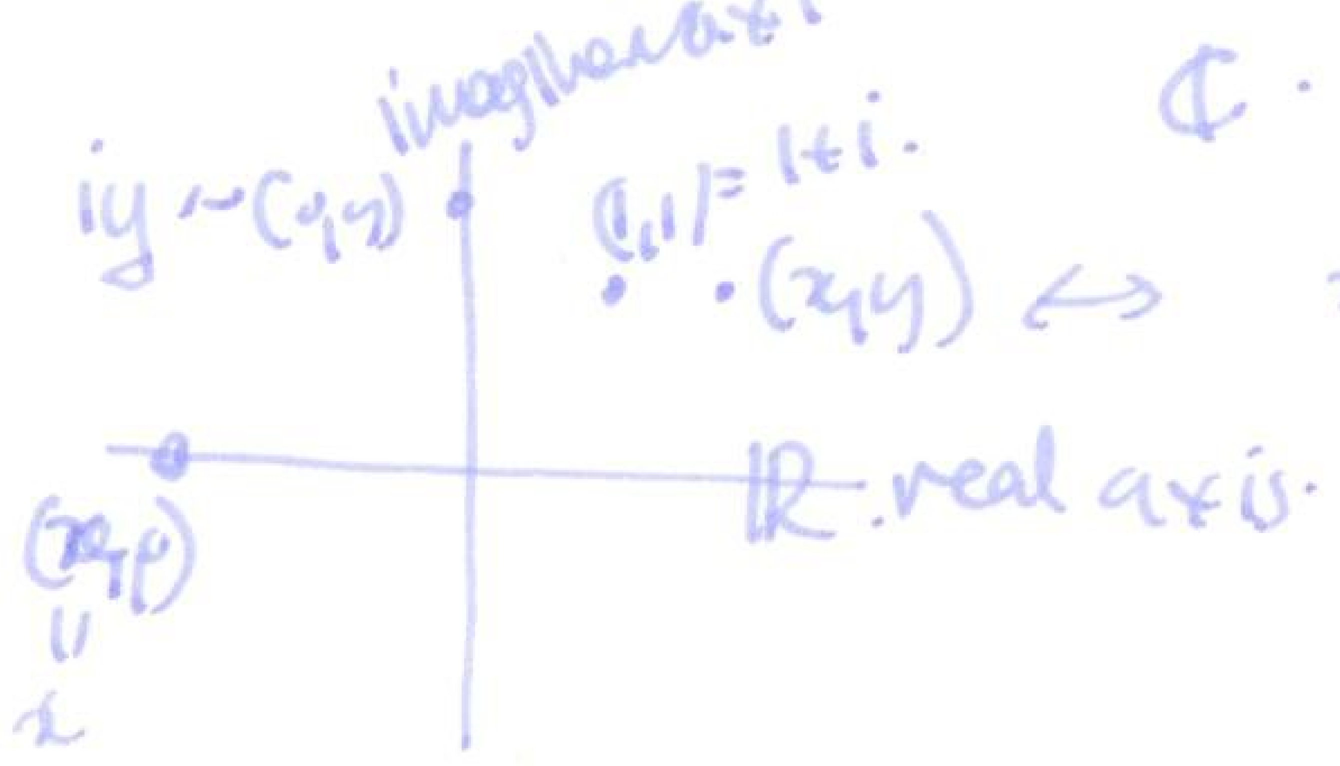
recall $i^2=-1$ $\quad a+bi \leftarrow$ complex numbers.

$$(a+bi)+(c+di) = a+c+(b+d)i$$

$$(a+bi)(c+di) = ac+adi+bci-bd$$
$$= (ac-bd)+(ad+bc)i.$$

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\,\frac{c-di}{c-di} = \frac{ac+bd+i(bc-ad)}{c^2+d^2}.$$

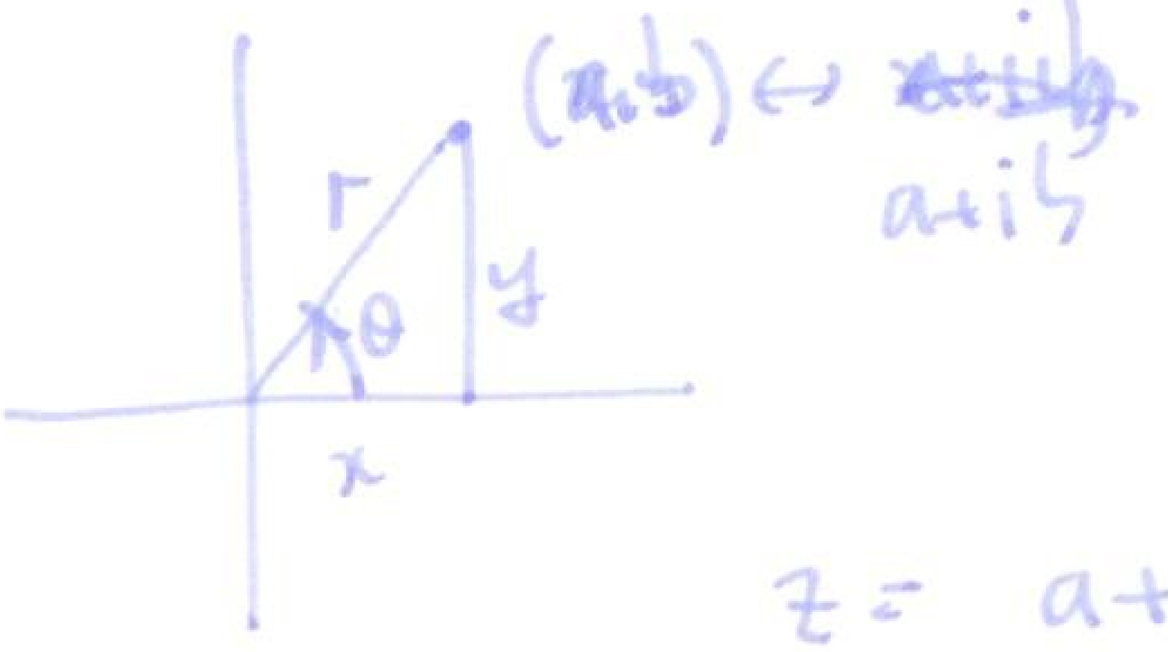
pictures $\quad \mathbb{C}$.



absolute value: $|z| = |a+bi| = \sqrt{a^2+b^2}$ (distance from 0 in $\mathbb{C}$)

Example $\quad |3+4i|=5 \qquad |-2-i| = \sqrt{5}. \qquad |\tfrac{4}{5}i| = \tfrac{4}{5}.$

trig notation for complex numbers.

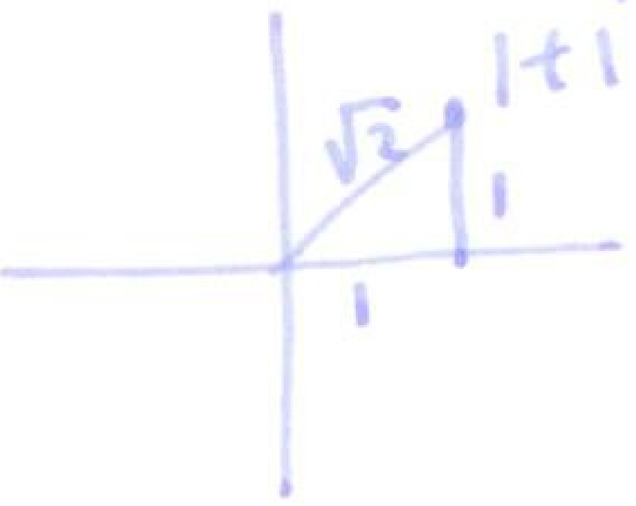


$r = |z| = \sqrt{\cancel{x^2+y^2}\; a^2+b^2}.$

$a = r\cos\theta$
$b = r\sin\theta$

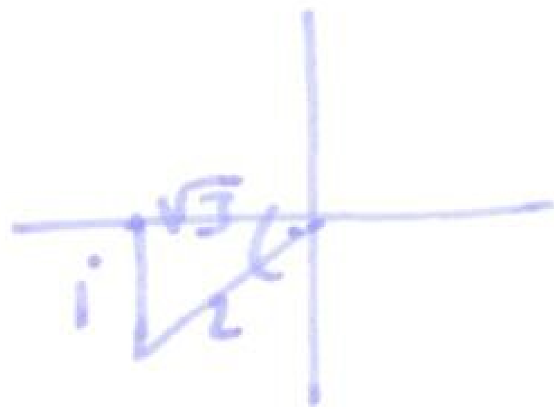
$$z = \underbrace{a+bi}_{\text{standard notation}} = \underbrace{r(\cos\theta + i\sin\theta)}_{\text{trig. notation}}.$$

examples $1+i$



$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$-\sqrt{3} - i$.



$= 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$.

multiplication and division

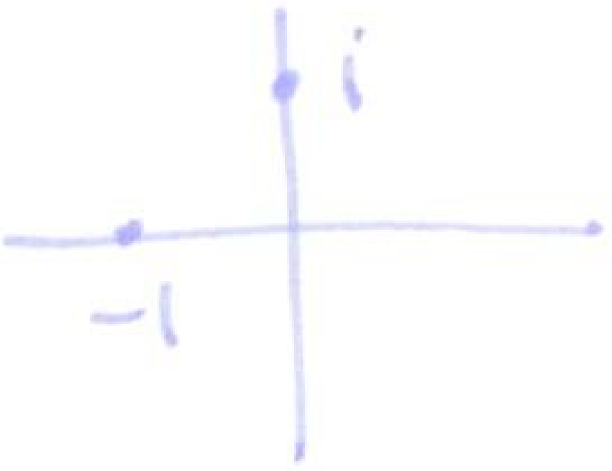
$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then $z_1 z_2 = \underbrace{r_1 r_2}_{\text{multiply arguments}}\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$ ⊛

add angles.

Example $i^2 = -1$.



$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

$i^2 = 1.1.\left(\cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right)$

$= 1(\cos(\pi) + i\sin(\pi))$.

Proof of ⊛

$z_1 = r\cos\theta_1 + ir\sin\theta_1$

$z_2 = r\cos\theta_2 + ir\sin\theta_2$.

$z_1 z_2 = (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2)$

$= r_1\cos\theta_1\, r_2\cos\theta_2 + r_1\cos\theta_1\, ir_2\sin\theta_2 + ir_1\sin\theta_1\, r_2\cos\theta_2 + i^2 r_1\sin\theta_1\, r_2\sin\theta_2$

$= r_1 r_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\right)$.

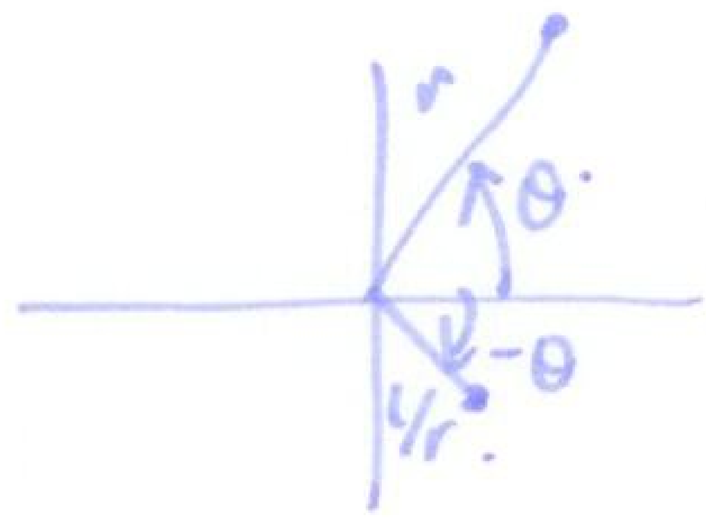
$= r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$. $\square$.

Division $\dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \dfrac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$.

note $\dfrac{1}{r(\cos\theta + i\sin\theta)} \cdot \dfrac{r(\cos\theta - i\sin\theta)}{r(\cos\theta - i\sin\theta)} = \dfrac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)}$

$= \dfrac{1}{r}(\cos\theta - i\sin\theta)$.

$= \dfrac{1}{r}(\cos(-\theta) + i\sin(-\theta))$.



Powers

$z = r(\cos\theta + i\sin\theta)$

$z^2 = z.z = r(\cos\theta + i\sin\theta).r(\cos\theta + i\sin\theta)$

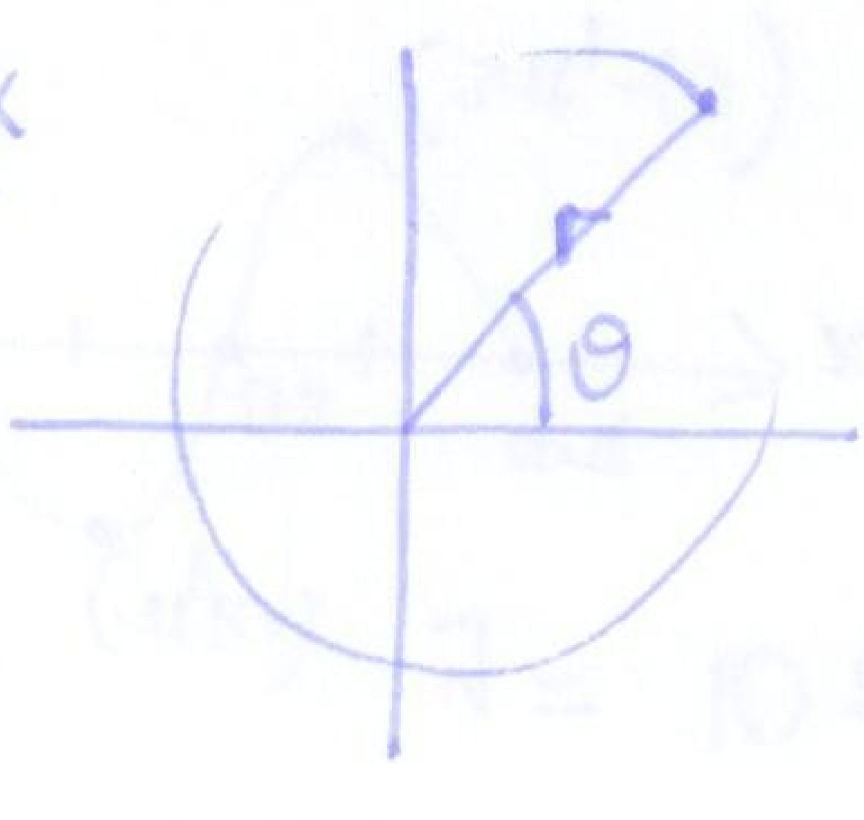
$= r^2(\cos(\theta+\theta) + i\sin(\theta+\theta)) = r^2(\cos 2\theta + i\sin 2\theta)$.

$= r^2(\cos^2\theta - \sin^2\theta + i\,2\sin\theta\cos\theta)$.

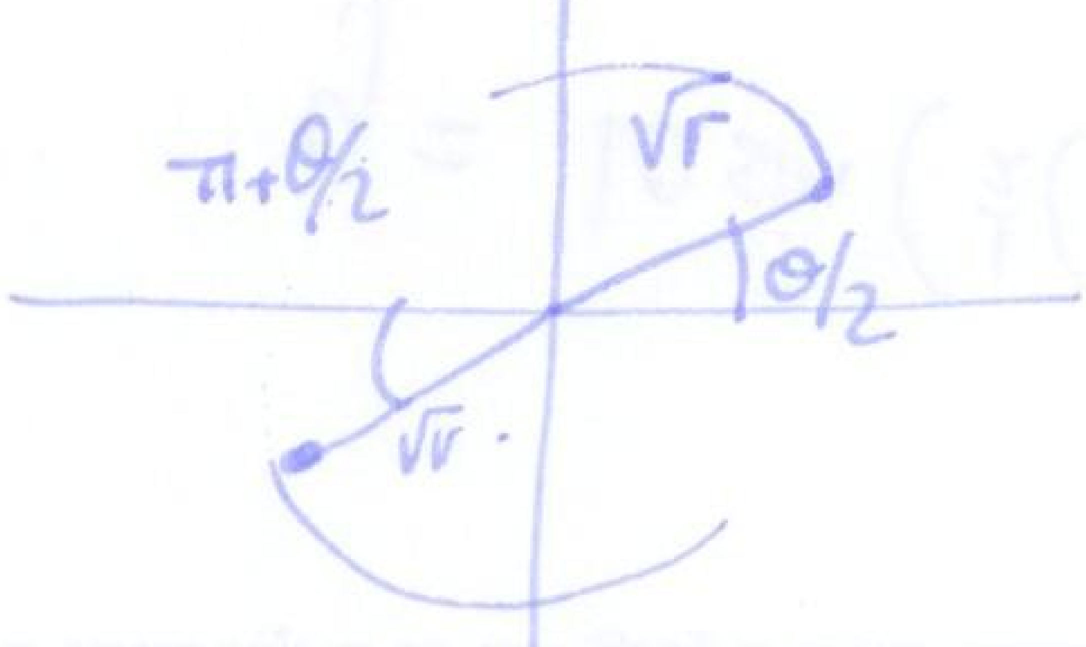
similar z^3: $r^3(\cos 3\theta + i\sin 3\theta) = r^3\left(\cos^3\theta + 2\cos\theta\sin^2\theta + i(2\cos^2\theta\sin\theta - \sin^3\theta)\right)$.

De Moivre's theorem $\left(r(\cos\theta + i\sin\theta)\right)^n = r^n(\cos n\theta + i\sin n\theta)$.

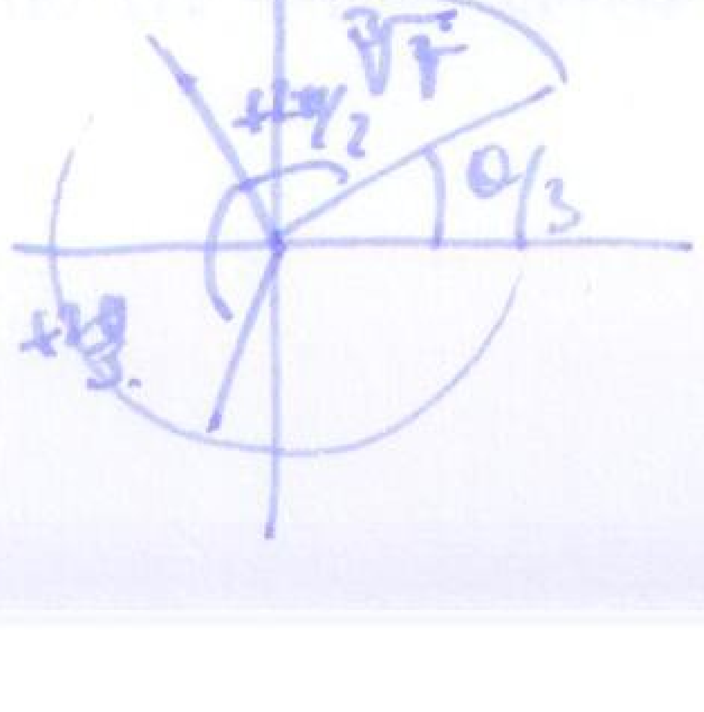
Roots of complex



square roots.



cube roots.



etc..

do. +1, -1, i.

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the nth roots of the complex number $r(\cos\theta + i\sin\theta)$ are given by: $r^{1/n}\left(\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)$

There are <u>n</u> of them.