

§8.3 Complex numbers

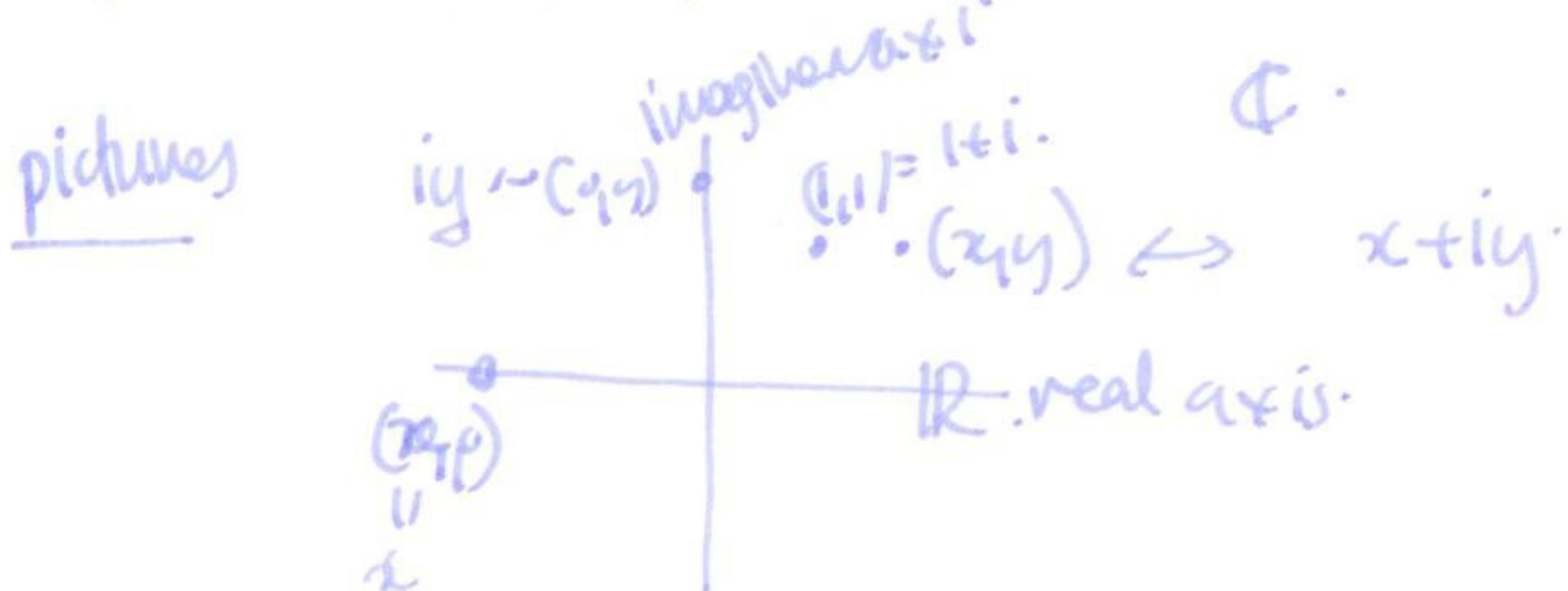
(54a)

recall $i^2 = -1$ $a+bi \leftarrow$ complex numbers.

$$(a+bi) + (c+di) = a+c + (b+d)i$$

$$\begin{aligned} (a+bi)(c+di) &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

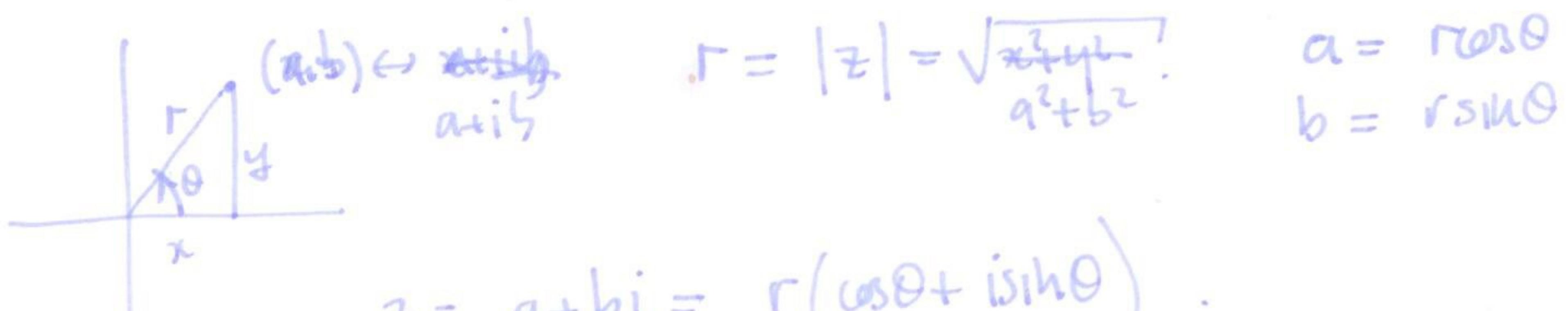
$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{\overline{c-di}}{\overline{c-di}} = \frac{ac+bd+i(c-a-d)}{c^2+d^2}.$$



absolute value: $|z| = |a+bi| = \sqrt{a^2+b^2}$ (distance from 0 in \mathbb{C})

Example $|3+4i|=5$ $|-2-i|=\sqrt{5}$ $\left|\frac{4}{5}i\right|=\frac{4}{5}$

Trig notation for complex numbers:



$$z = a+bi = r(\cos \theta + i \sin \theta)$$

standard notation trig. notation.

$$(3) \text{ Let } z_1 = 1+2i, z_2 = 3+4i, z_3 = -2+2i. \text{ Then } z_1 z_2 z_3 = \dots$$

Example $|1+i| = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$ 54b

$= -\sqrt{3} - i$

$|-\sqrt{3} - i| = 2 \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right)$

multiplication and division

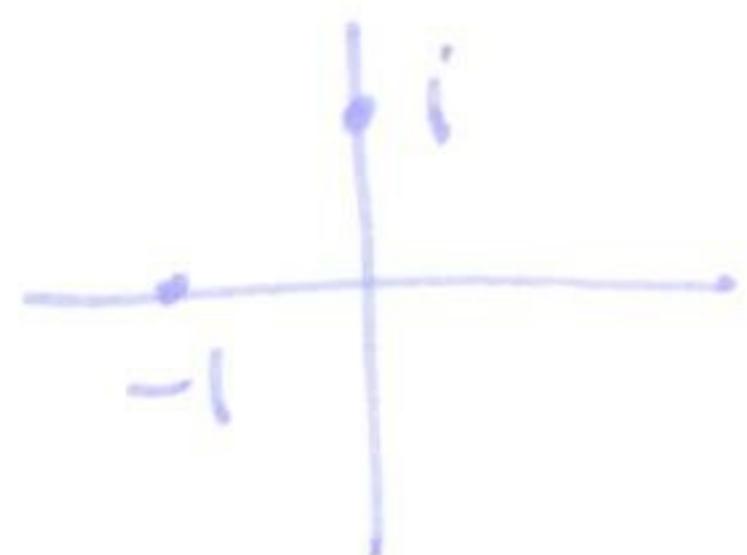
$$z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

then $z_1 z_2 = \underbrace{r_1 r_2}_{\text{multiply argument}} \left(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right)$ ⊕

\uparrow add angles.

Example $i^2 = -1$.



$$i = 1 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right).$$

$$\begin{aligned} i^2 &= 1 \cdot 1 \left(\cos \left(\frac{\pi}{2} + \frac{\pi}{2} \right) + i \sin \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \right) \\ &= 1 \left(\cos(\pi) + i \sin(\pi) \right). \end{aligned}$$

Proof of ⊕ $z_1 = r_1 \cos \theta_1 + i r_1 \sin \theta_1$
 $z_2 = r_2 \cos \theta_2 + i r_2 \sin \theta_2$

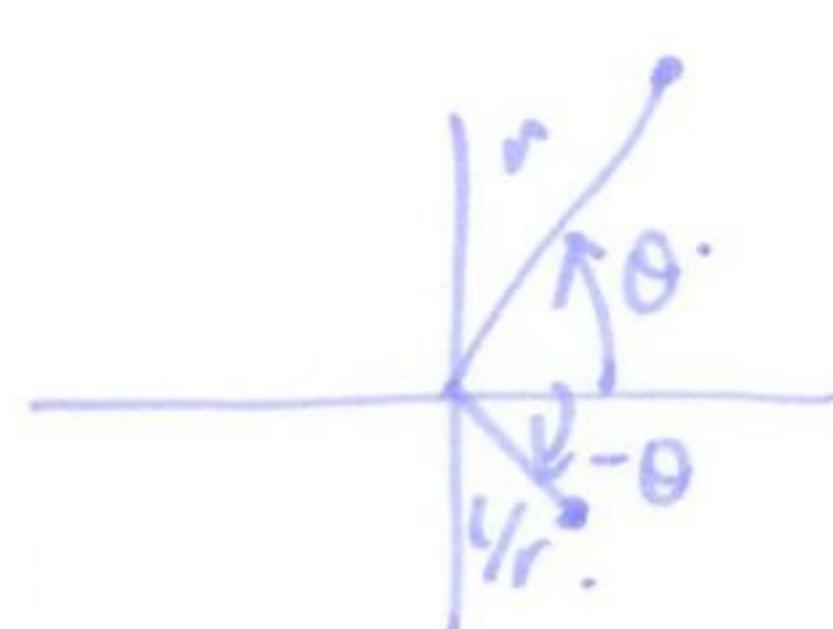
$$\begin{aligned} z_1 z_2 &= (r_1 \cos \theta_1 + i r_1 \sin \theta_1)(r_2 \cos \theta_2 + i r_2 \sin \theta_2) \\ &= r_1 \cos \theta_1 r_2 \cos \theta_2 + r_1 \cos \theta_1 i r_2 \sin \theta_2 + i r_1 \sin \theta_1 r_2 \cos \theta_2 + i^2 r_1 \sin \theta_1 r_2 \sin \theta_2 \\ &= r_1 r_2 \left(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i (\cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1) \right). \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) . \quad \square. \end{aligned}$$

Division

$$\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

Note

$$\begin{aligned} & \frac{1}{r(\cos \theta + i \sin \theta)} \cdot \frac{r(\cos \theta - i \sin \theta)}{r(\cos \theta - i \sin \theta)} = \\ &= \frac{1}{r} (\cos \theta - i \sin \theta) \\ &= \frac{1}{r} (\cos(-\theta) + i \sin(-\theta)) \end{aligned}$$

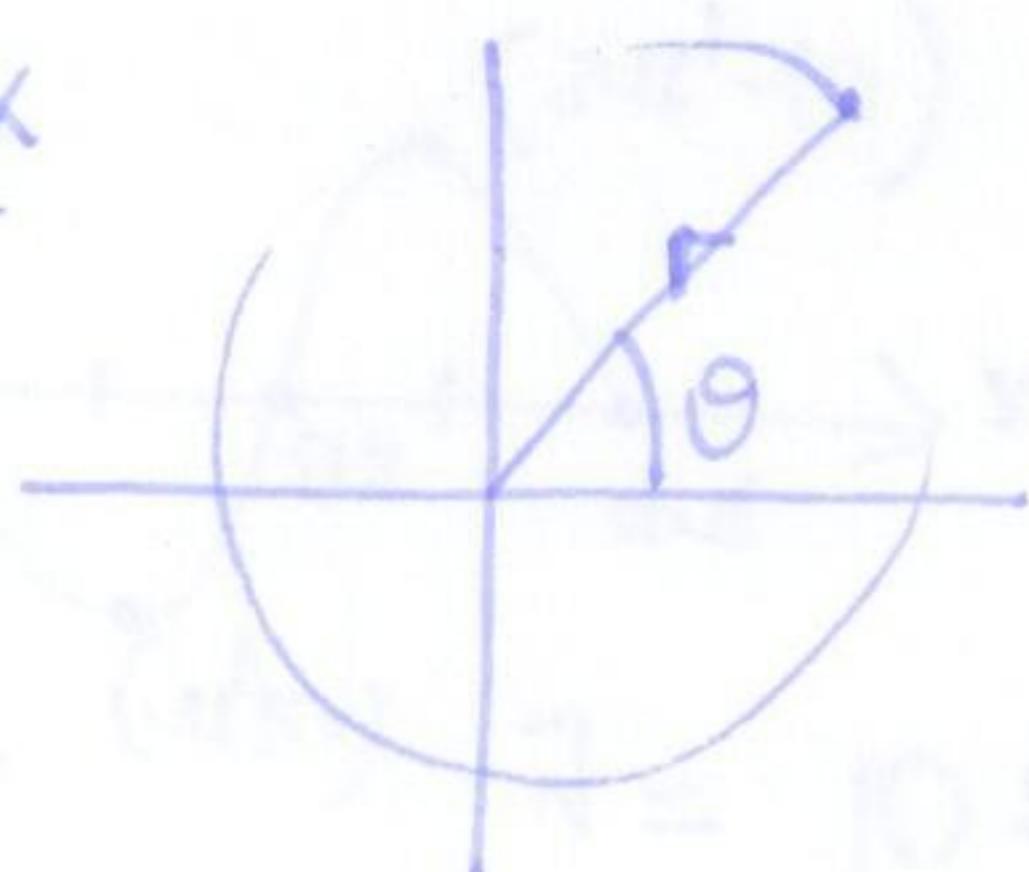
Powers

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) = r^2 (\cos^2 \theta - \sin^2 \theta + i(2 \sin \theta \cos \theta)) \\ z^2 &= z \cdot z = r(\cos \theta + i \sin \theta) \cdot r(\cos \theta + i \sin \theta) \\ &= r^2 (\cos(\theta + \theta) + i \sin(\theta + \theta)) = r^2 (\cos 2\theta + i \sin 2\theta) \end{aligned}$$

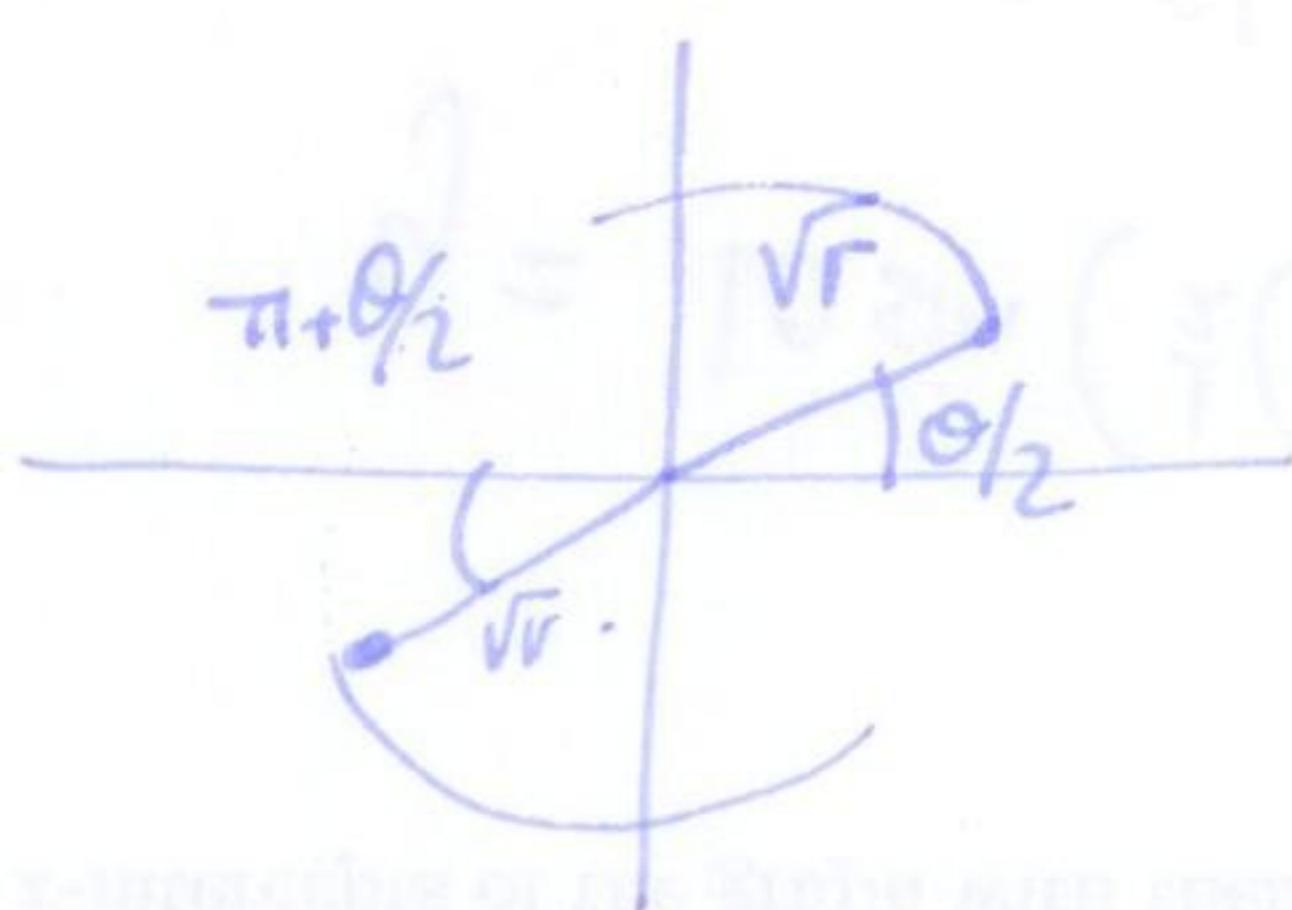
similarly $z^3 = r^3 (\cos 3\theta + i \sin 3\theta) = r^3 (\cos^3 \theta + 3 \cos \theta \sin^2 \theta + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta))$

De Moivre's theorem

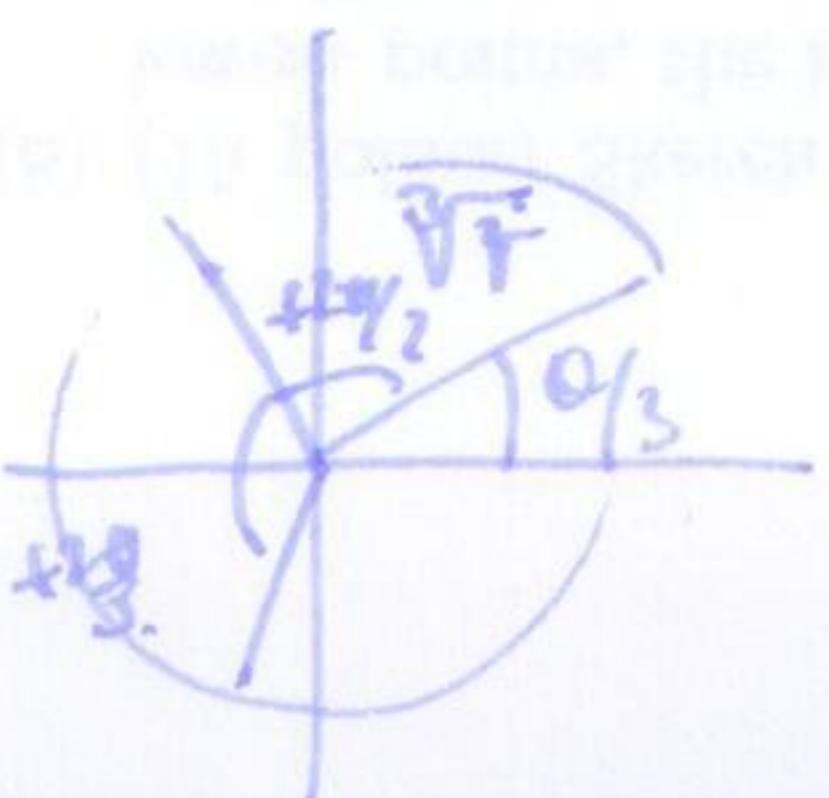
$$(r(\cos \theta + i \sin \theta))^n = r^n (\cos(n\theta) + i \sin(n\theta))$$

Roots of complex

square roots



cube roots.



etc..

do. $\pm 1, -1, i$.

The n th roots of the complex number $r(\cos\theta + i\sin\theta)$
are given by: $\sqrt[n]{r} \left(\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) \right)$

There are n of them.

Explain the following terms:
 (a) $\sin(x)$, (b) $\cos(x)$, and (c) $\cos(x + \pi)$. What is
 (d) the modulus of $i\sqrt{3}$? Explain what amplitude corresponds to this value.
 (e) (if possible) If $\cos(x) = -\frac{1}{2}$ (x in radians), then $x = \frac{2\pi}{3}$ (x in degrees).