Important facts:

- if a function is one-to-one then it has an inverse f^{-1}!
- if f^{-1} exists: the domain of f^{-1} is the range of f $\quad \mathbb{R} \xrightarrow{f} \mathbb{R}$, $\xleftarrow{f^{-1}}$.
  range of f^{-1} domain of f
- if $f: \mathbb{R} \to \mathbb{R}$ is strictly increasing or strictly decreasing, then it is one-to-one 1-1.

Example $\quad f(x) = ~~ex~~ \; 3x-4$

Q: is $f(x)$ one-to-one?

suppose $f(a) = f(b) \Rightarrow 3a-4 = 3b-4$.

$3a = 3b$.

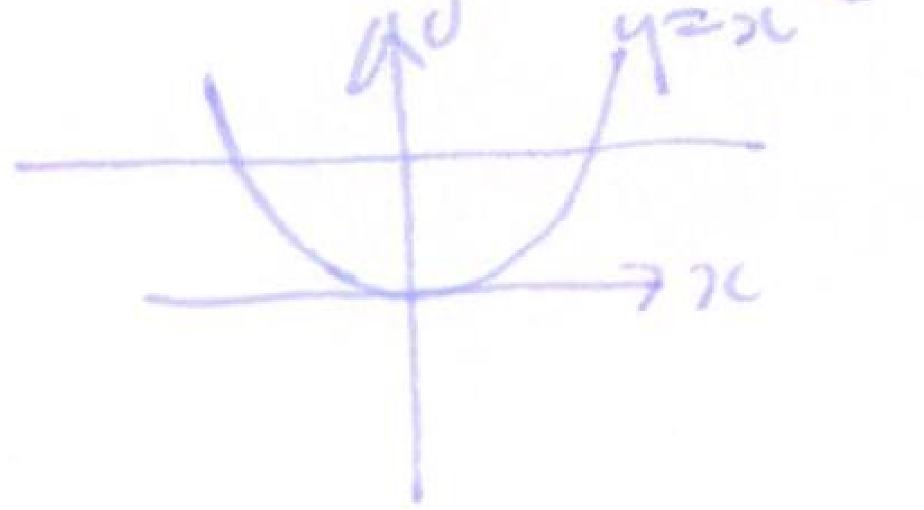
$a = b$ so f one-to-one.

so $f(x) = 3x-4$ has an inverse.

Horizontal line test: a function $f: \mathbb{R} \to \mathbb{R}$ has an inverse / is one-to-one if every horizontal line hits the graph exactly once.

Example

$f(x) = x^2$



Q is $f(x)$ one-to-one?

suppose $f(a) = f(b) \Rightarrow a^2 = b^2$.

$\Rightarrow \pm a = \pm b$

no.

How to find the inverse?

① algebraically $\quad f(x) = y$ solve for $x =$ something in y.

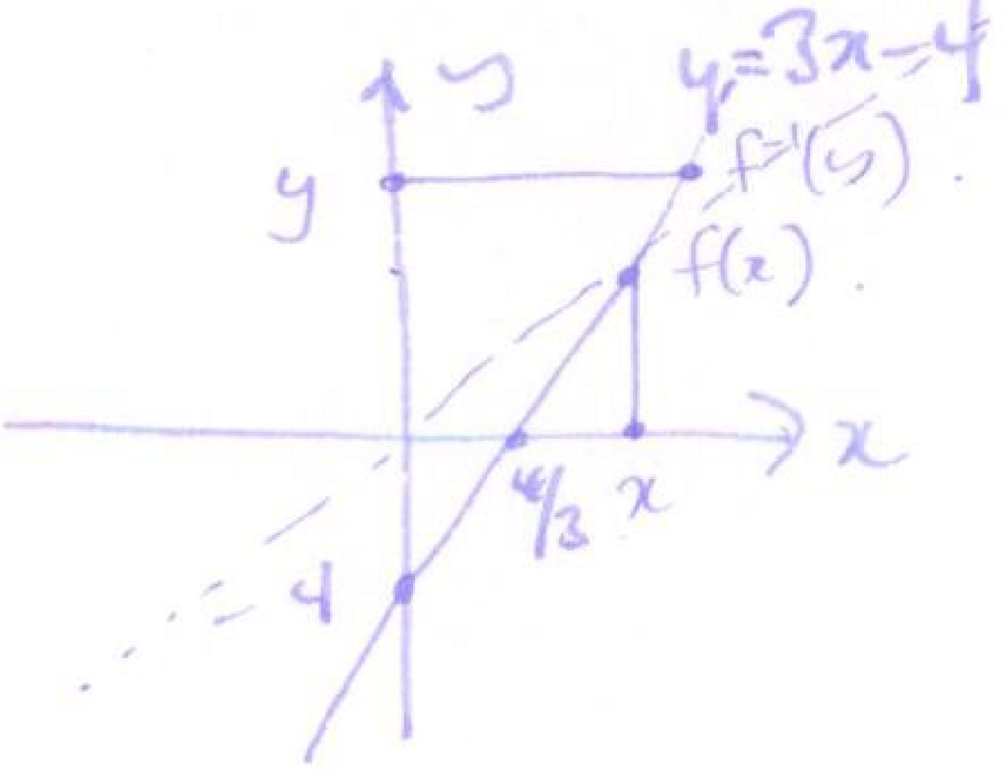
Example. $f(x) = 3x-4$. $\qquad y = 3x-4$.

$y+4 = 3x \qquad x = \dfrac{y+4}{3}$.

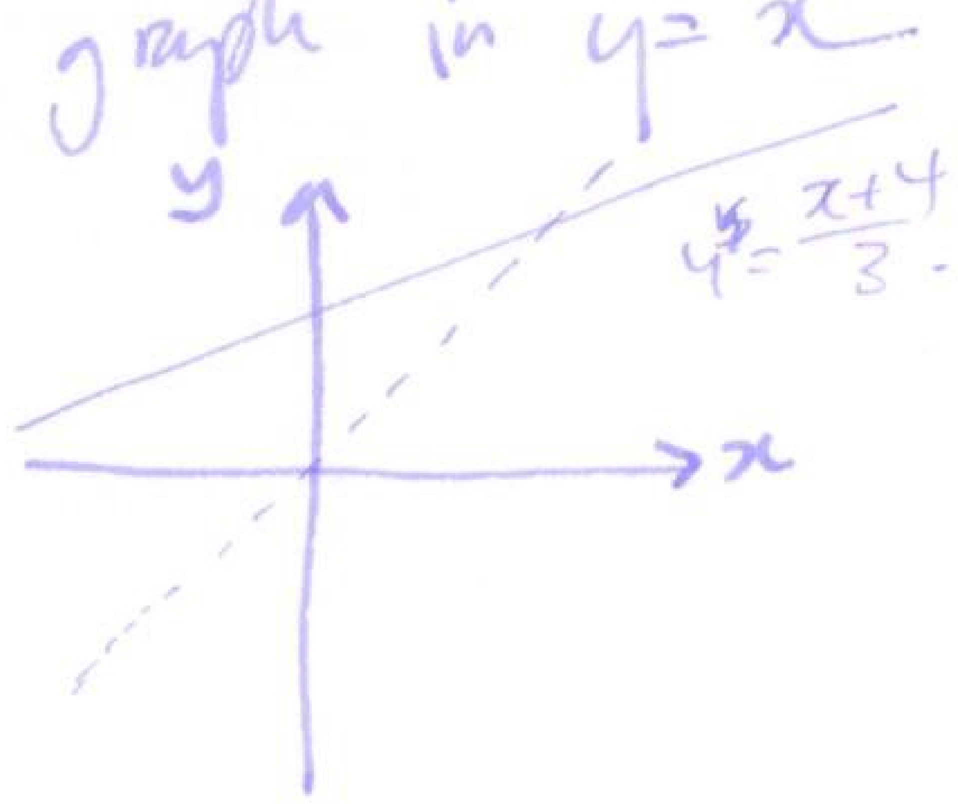
$f^{-1}(x) = \dfrac{x+4}{3}$

check! $f^{-1}(f(x)) = f^{-1}(3x-4) = \dfrac{(3x-4)+4}{3} = \dfrac{3x}{3} = x.$

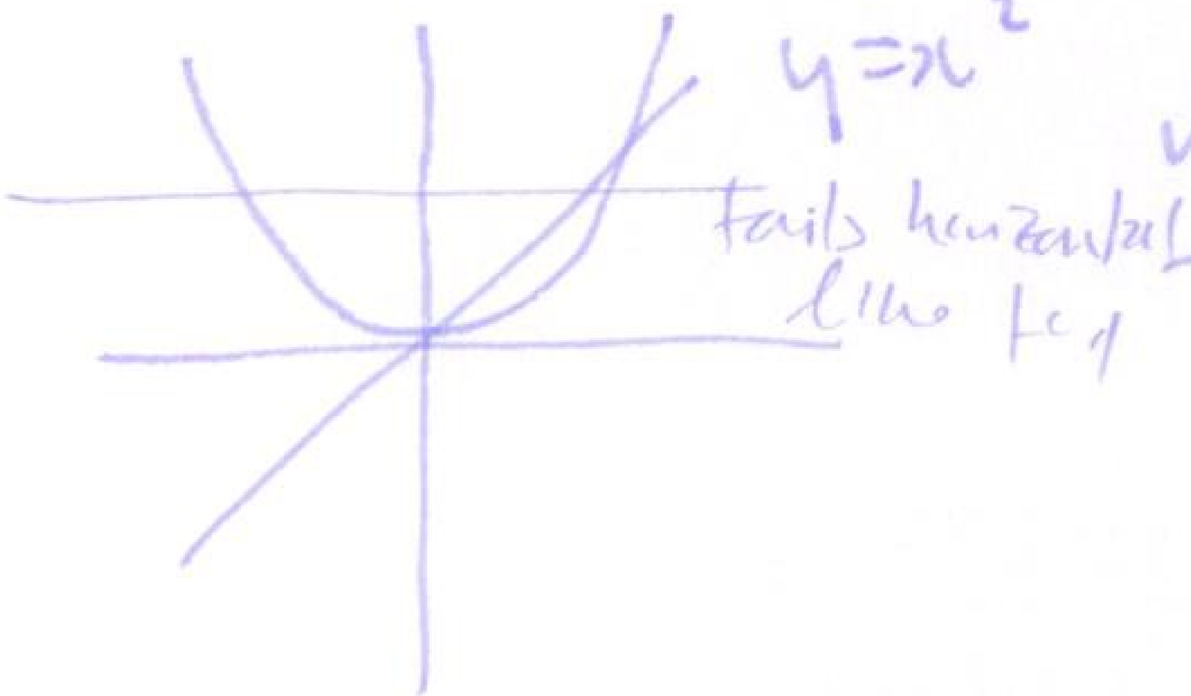
② graphically



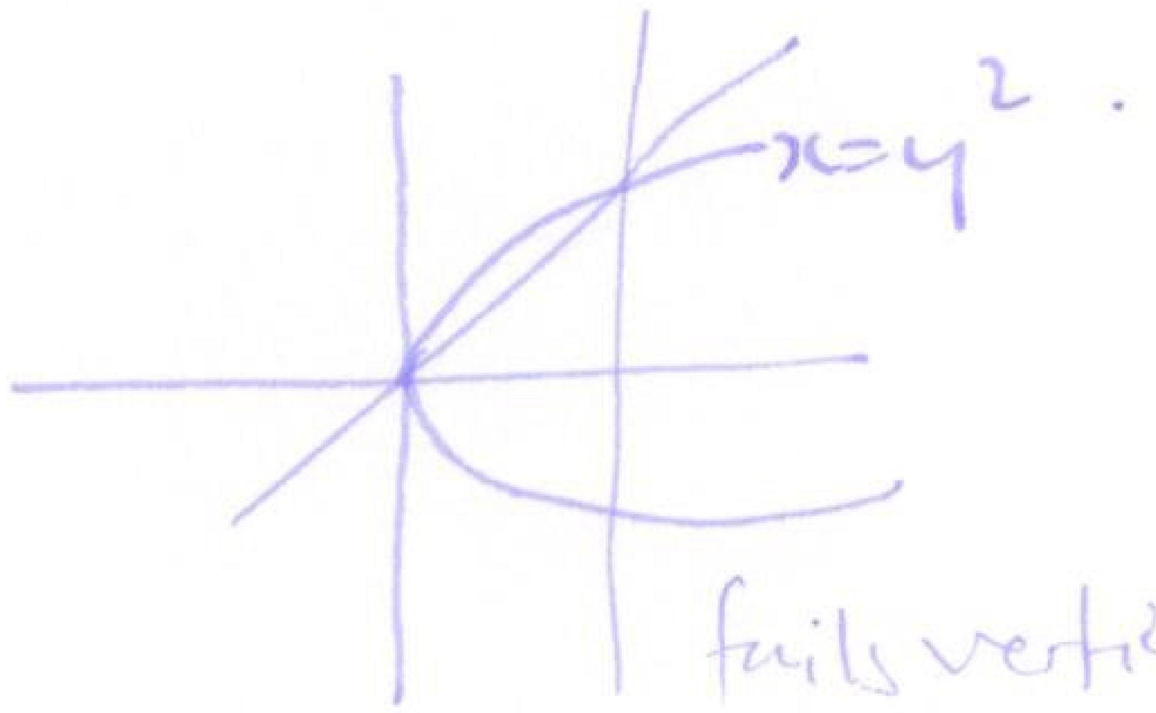
reflect graph in $y=x$.



observation: you can always reflect any graph in $y=x$.



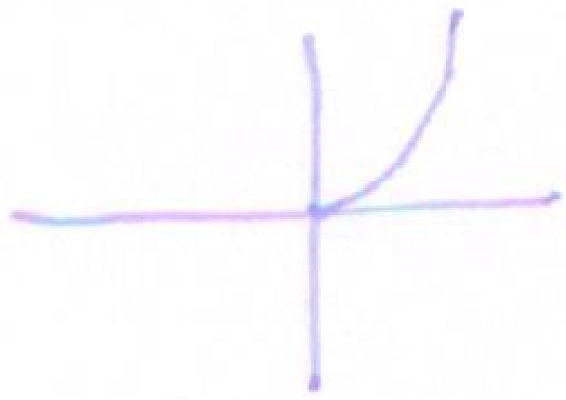
reflect

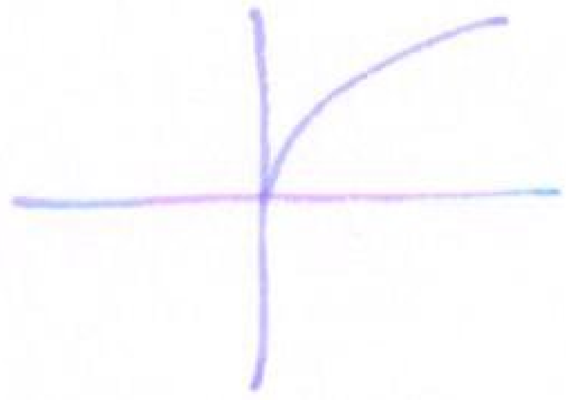


~~Example~~ Problem but I want a $\sqrt{x}$ function!

Solution: restrict domain until function one-to-one

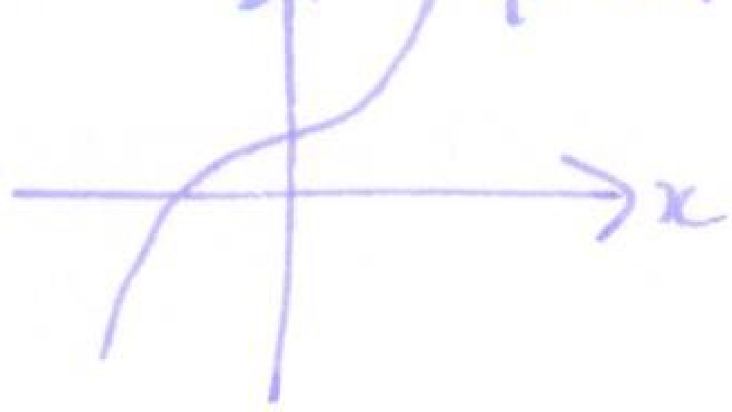
$y=x^2,\ x \geq 0$





has inverse $\sqrt{x}$.

Example $f(x) = x^3+1$



Q: one-to-one? suppose $f(a)=f(b)$.

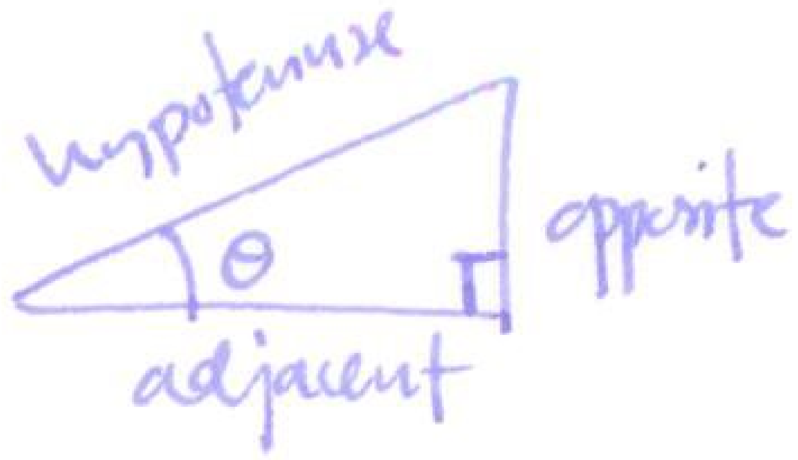
$a^3+1 = b^3+1 \Rightarrow a^3=b^3 \Rightarrow a=b.$ yes.

inverse: $y=x^3+1$ solve for x.

$y-1 = x^3$

$\sqrt[3]{y-1} = x$ so $f^{-1}(x) = \sqrt[3]{x-1}$.

§6.1 Right angled triangles



special functions defined in terms of a right angled triangle, the trigonometric functions.

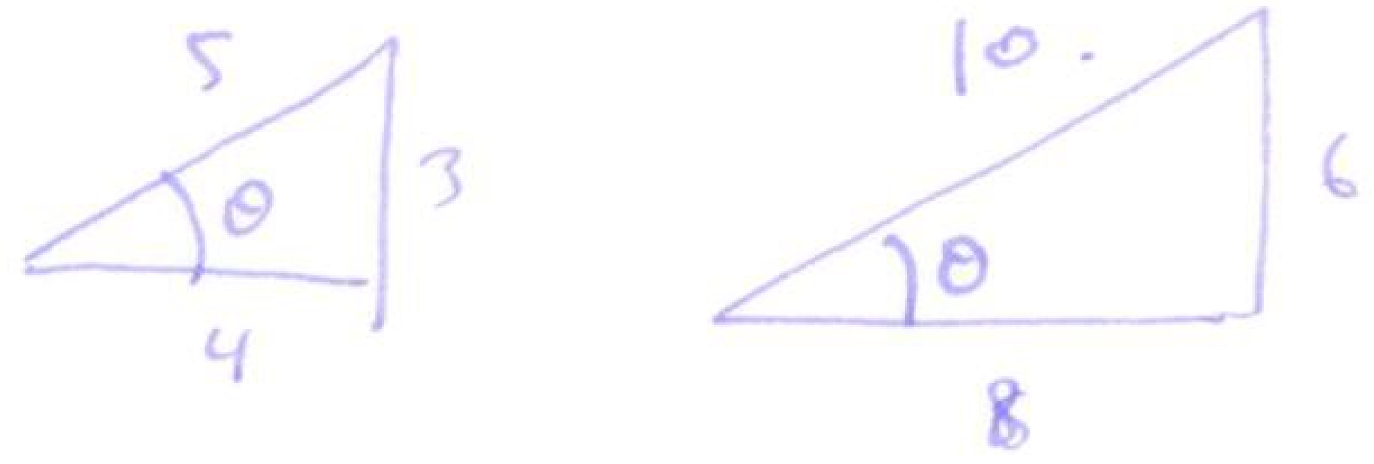
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{opp}}$$

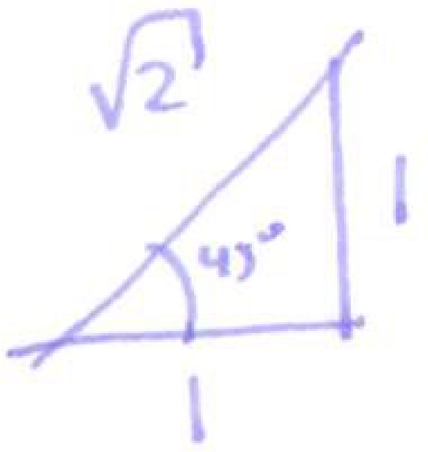
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \operatorname{cotan}(\theta) = \frac{1}{\tan\theta} = \frac{\text{adj}}{\text{opp}} = \frac{\cos\theta}{\sin\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

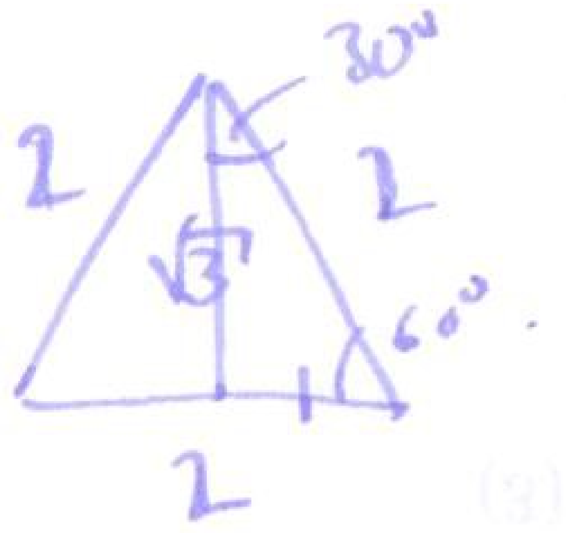
<u>note</u> these functions only depend on angles, so similar triangles gives same values.



Special values from special triangles

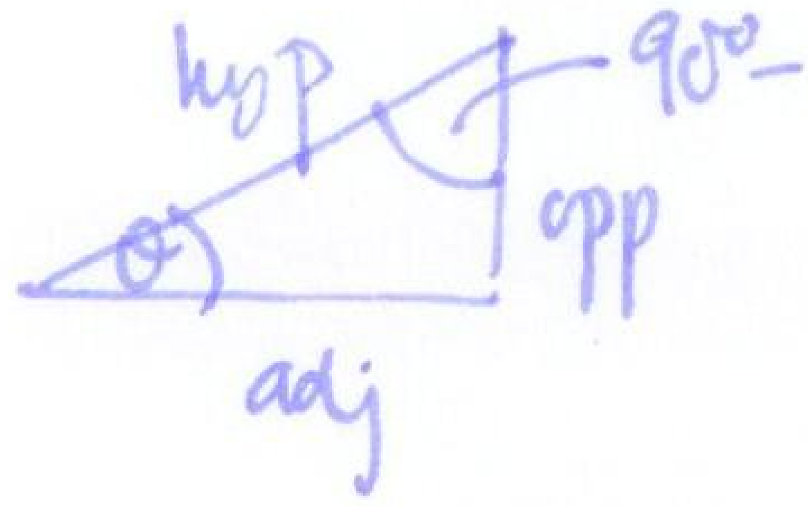


$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$
$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = 1.$$



$$\sin 30 = \tfrac{1}{2} \qquad \sin 60 = \tfrac{\sqrt{3}}{2}$$
$$\cos 30 = \tfrac{\sqrt{3}}{2} \qquad \cos 60 = \tfrac{1}{2}$$
$$\tan 30 = \tfrac{1}{\sqrt{3}} = \tfrac{\sqrt{3}}{3}. \qquad \tan 60 = \sqrt{3}.$$

Useful fact:

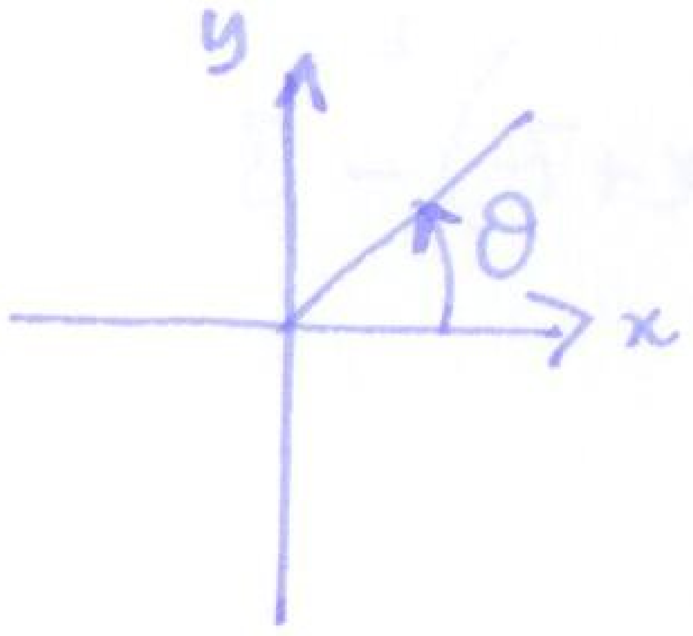


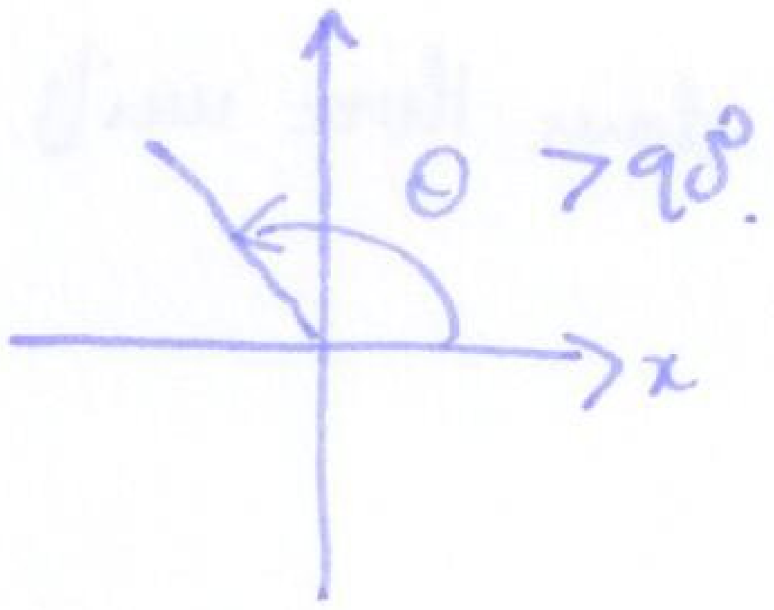
$$\sin\theta = \frac{opp}{hyp} = \cos(90^\circ-\theta).$$

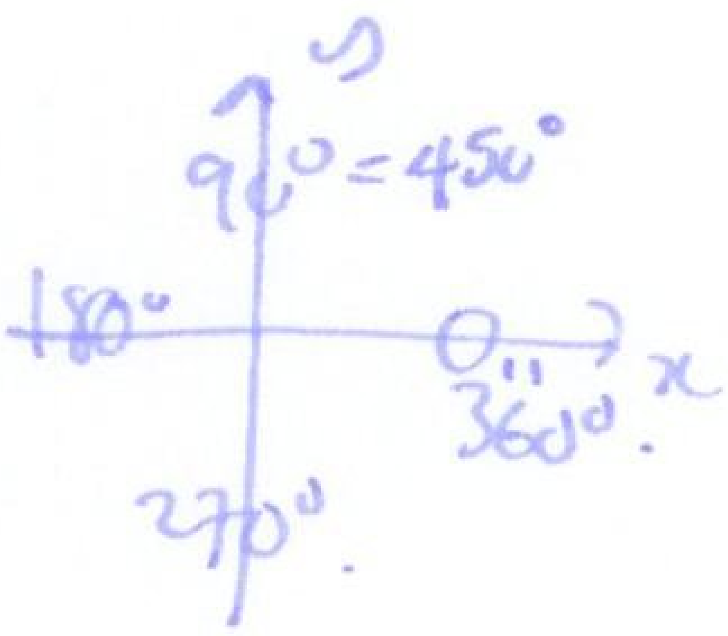
$$\cos\theta = \frac{adj}{opp} = \sin(90^\circ-\theta).$$

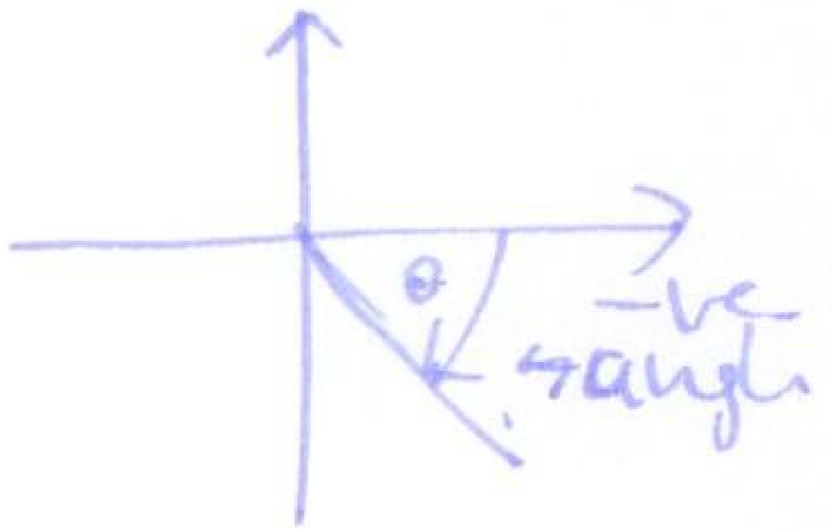
$$\tan\theta = \frac{opp}{adj} = \frac{1}{\tan(90^\circ-\theta)} = \cot(90^\circ-\theta).$$

§6.3 Trig functions for any angle



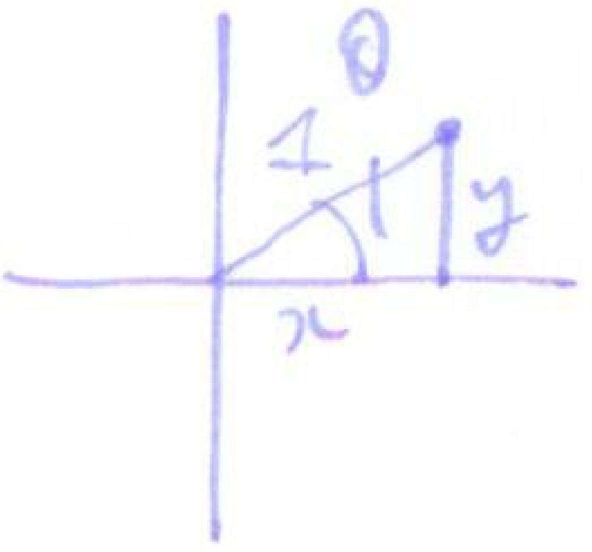






θ can be any real number.

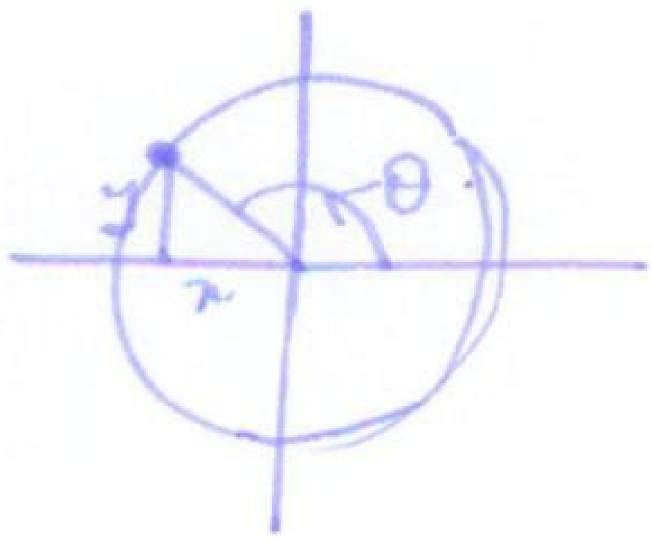
$$\theta = \theta + 360^\circ = \theta + 720^\circ = \cdots$$

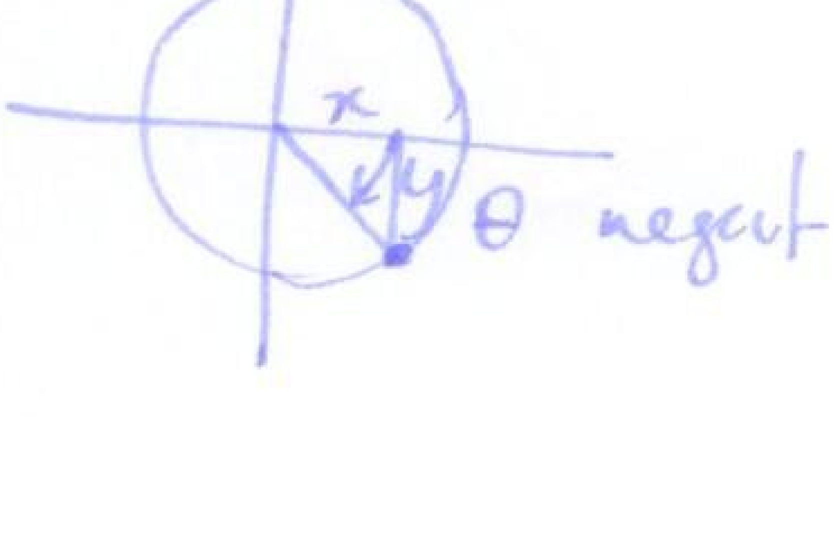


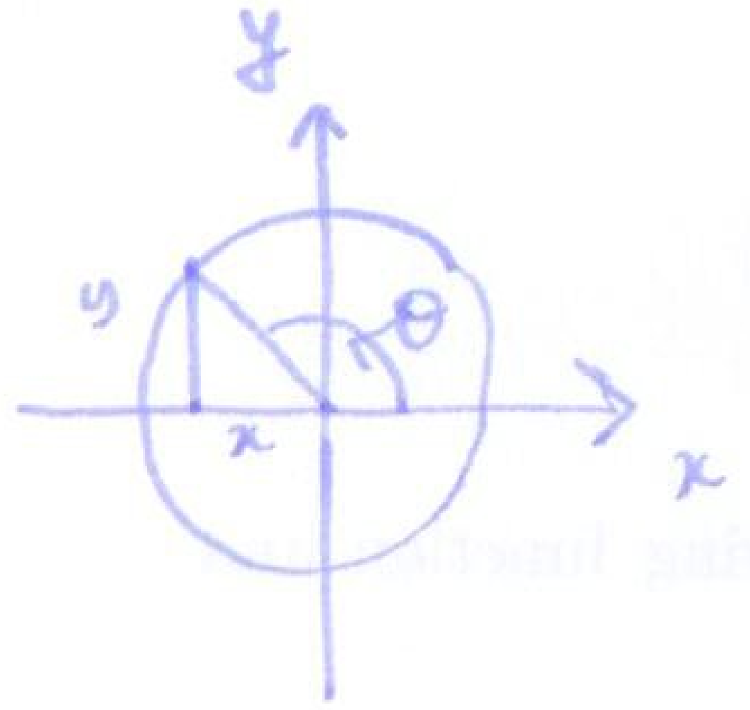
$$\sin\theta = \frac{opp}{hyp} = \frac{y}{1} \text{ coordinate.}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{x}{1} \quad x \text{ coordinate}$$

$$\tan\theta = \frac{opp}{adj} = \frac{y}{x}$$



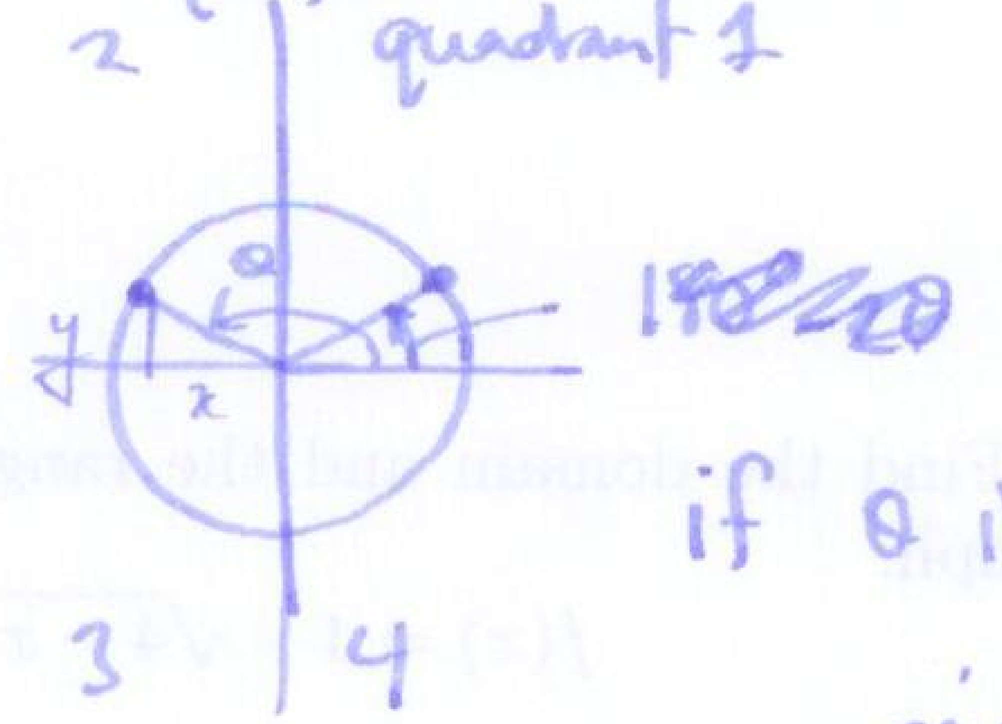




reflect :

reflect in y-axis.

quadrant 1

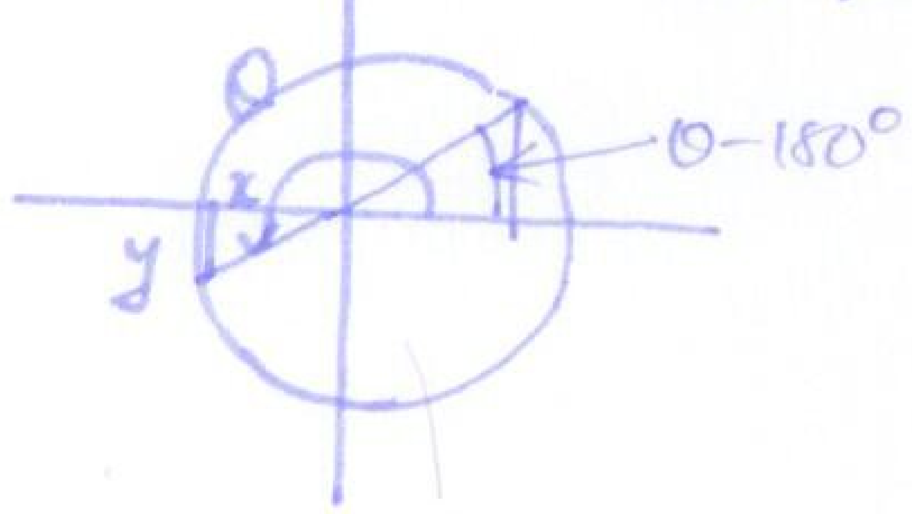


if θ in quadrant 2 :

$$\sin\theta = +\sin(180° - \theta).$$
$$\cos\theta = -\cos(180° - \theta).$$
$$\tan\theta = -\tan(180° - \theta).$$

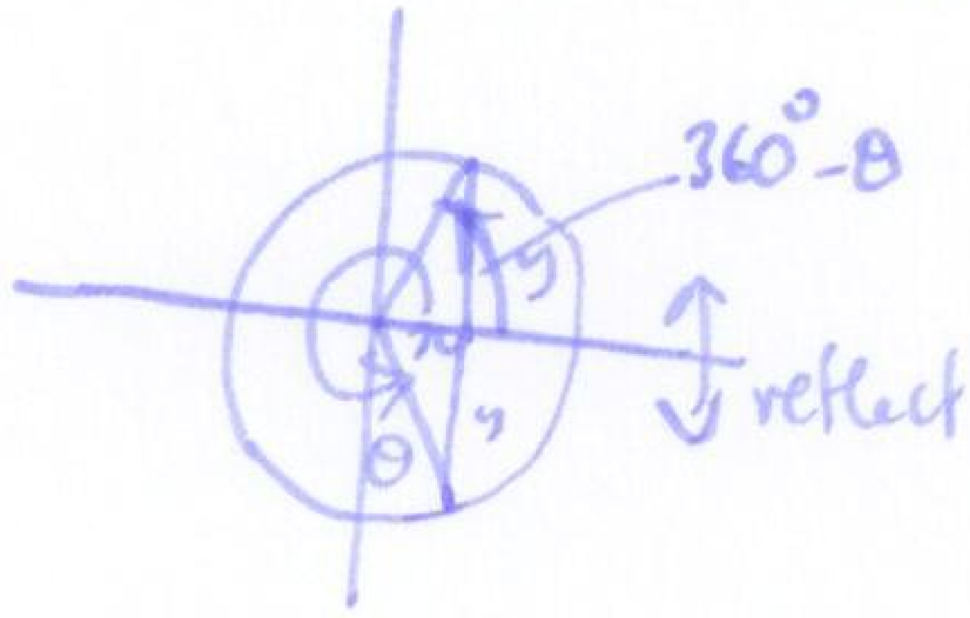
if θ in quadrant 3 :

← half turn



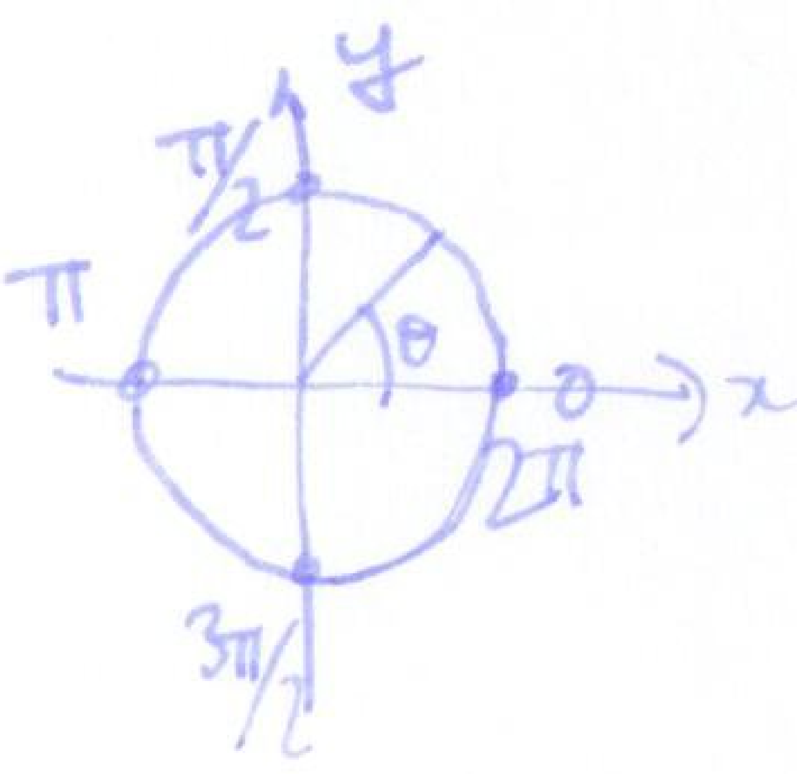
$$\sin\theta = -\sin(\theta - 180°).$$
$$\cos\theta = -\cos(\theta - 180°)$$
$$\tan\theta = \tan(\theta - 180°).$$

in quadrant 4:



$$\sin\theta = -\sin(360° - \theta).$$
$$\cos\theta = \cos(360° - \theta)$$
$$\tan\theta = -\tan(360 - \theta)$$

§6.4 Radians + arc length



unit circle has ~~radius~~ circumference 2π.

radians measure angle by distance around unit circle.

i.e $90° = \pi/2$. $\theta° = \frac{2\pi\theta}{360}$

$180° = \pi$ etc...