same factors may be the same, eg $x^4 = (x-0)(x-0)(x-0)(x-0)$.

<u>useful fact</u> if you know the zeros you can write down the polynomial

<u>example</u> degree 3 polynomial $p(x)$ has roots $2, 1+i, 1-i$

$$p(x) = (x-2)(x-1-i)(x-1+i)$$

$$= (x-2)\left(x^2 + x(-1-i-1+i) + \underbrace{(-1-i)(-1+i)}_{1+1}\right).$$

$$= (x-2)(x^2-2x+2)$$

$$= \begin{array}{l} x^3 - 2x^2 + 2x \\ \quad\; - 2x^2 + 4x - 4 \end{array} = x^3 - 4x^2 + 6x - 4.$$

<u>useful fact</u> • if a polynomial has <u>real</u> coefficients, then any complex roots come in conjugate pairs $a+bi$ and $a-bi$.

• if a polynomial has rational coefficients there is an ~~Effective~~ algorithm to find any rational roots (but there might not be any)

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

all coefficients integers. if p/q is a factor of $p(x)$ (p, q relatively prime) then p divides a_0 and q divides a_n.

<u>Example</u> $f(x) = 3x^4 - 11x^3 + 10x - 4 = (x+1)(x-\frac{2}{3})(3x^2-12x+6)$

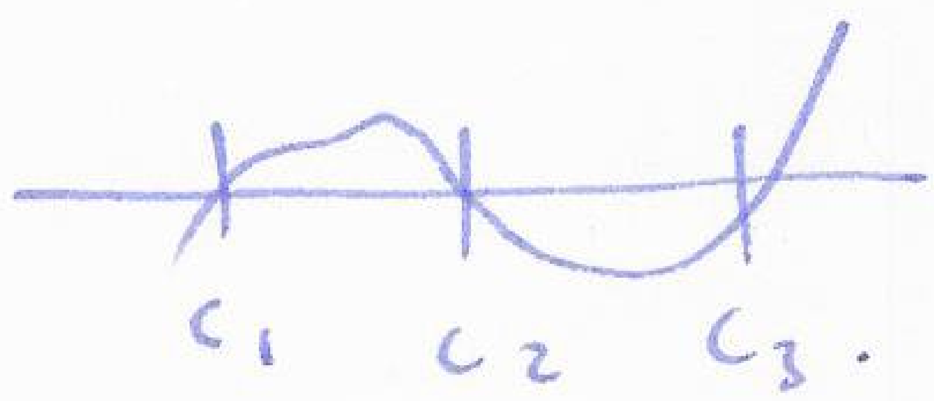
<u>Q</u>: are there any rational roots? check $\frac{p}{q}$ p can be $\pm1 \;\pm2 \;\pm4$

q $\pm1 \;\pm3$.

what happens near a zero?

$p(x) = (x-c_1)(x-c_2)\cdots(x-c_n)$.

all c_i different



changes sign at c_i

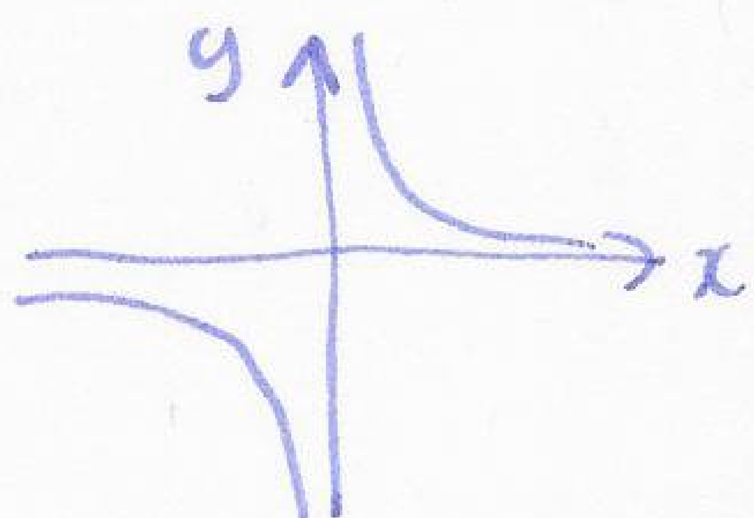
unless $(x-c_i)^n$ occurs with an even power

Descartes rule of signs?

§4.5 Rational functions

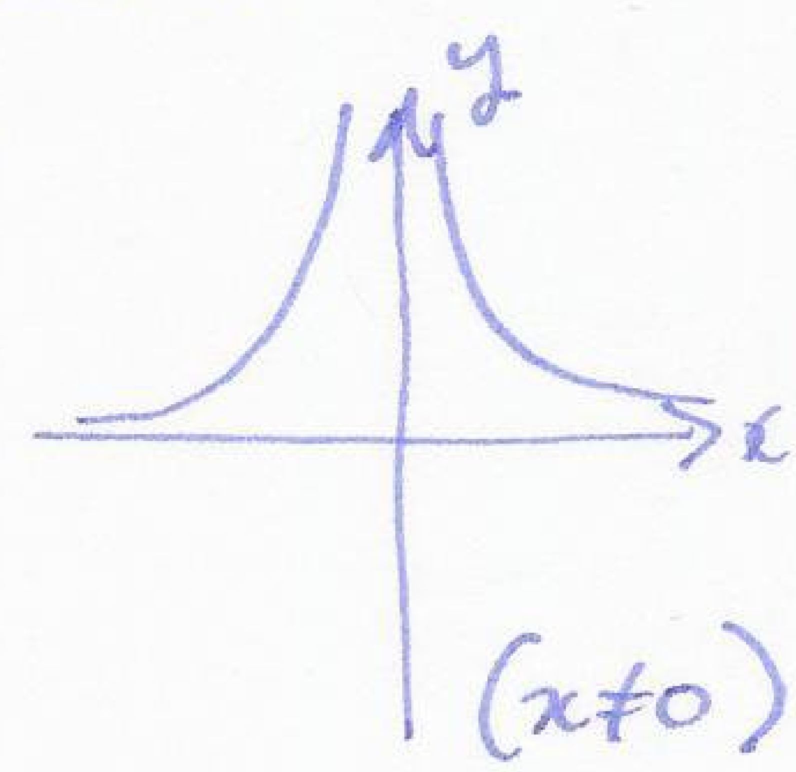
A rational function is a quotient of two polynomials. $f(x) = p(x)/q(x)$

Examples $f(x) = 1/x$



$(x \neq 0)$

$1/x^2$

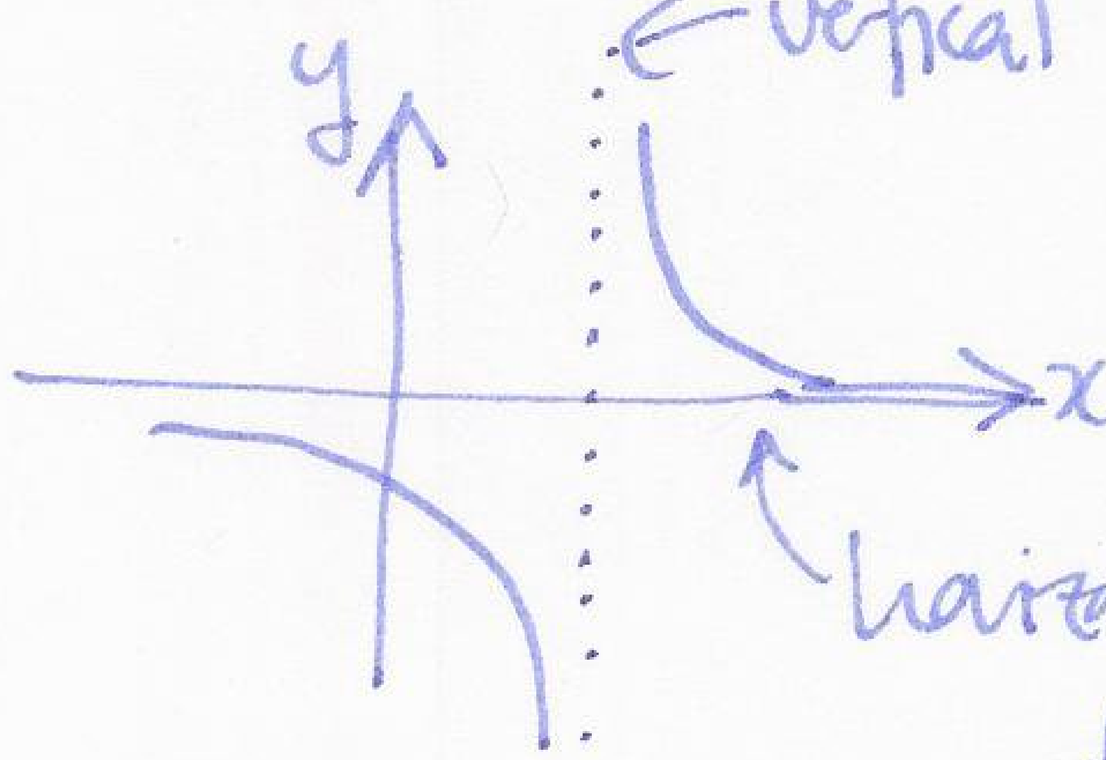


$(x \neq 0)$.

domain: when $q(x) \neq 0$.

Example $f(x) = \dfrac{1}{x-2}$

domain $x \neq 2$.

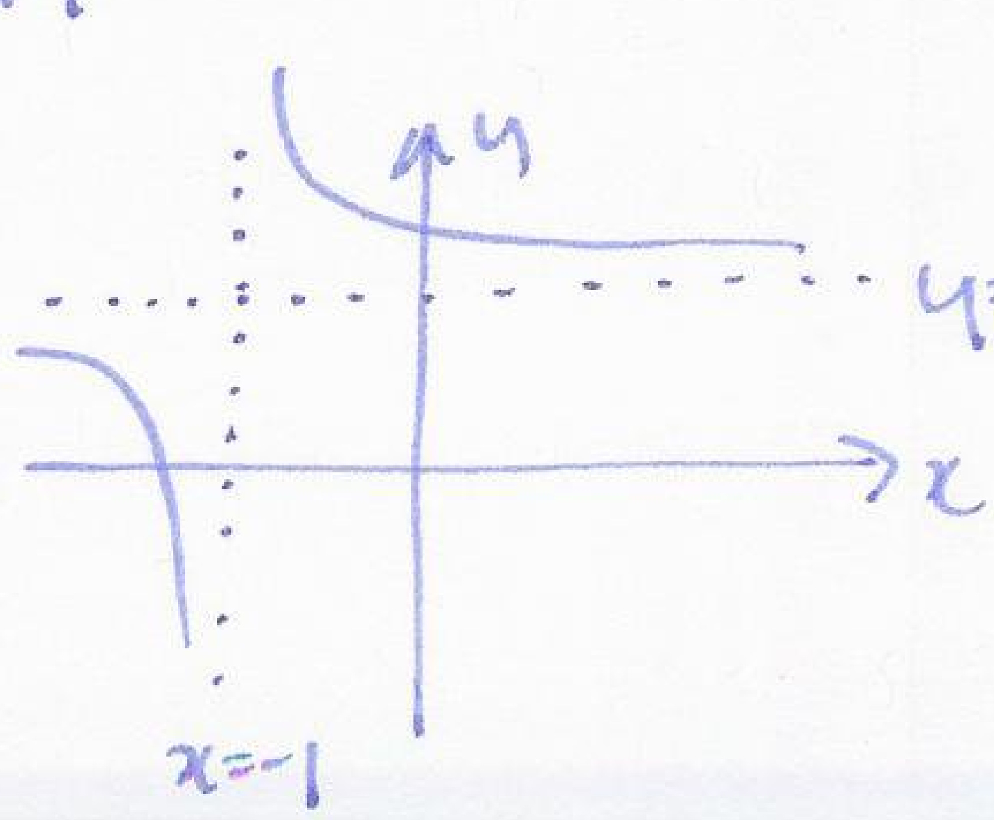


vertical asymptote at $x=2$

horizontal asymptote at $y=0$

Example $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$

$$\begin{array}{r|l} & 1 \\ \hline x+1 & x+2 \\ & x+1 \\ \hline & 1. \end{array}$$



vertical asymptote $x=-1$

horizontal asymptote $y=1$.

Finding horizontal asymptotes.

$f(x) = \dfrac{p(x)}{q(x)}$ what happens when $|x|$ is large?

i.e. $x \to +\infty$
$x \to -\infty$.

if $\deg(p) > \deg(q)$ then $|f(x)| \to \infty$ as $|x| \to \infty$.
(don't know about signs)

if $\deg(p) < \deg(q)$ then $|f(x)| \to 0$ as $|x| \to \infty$.

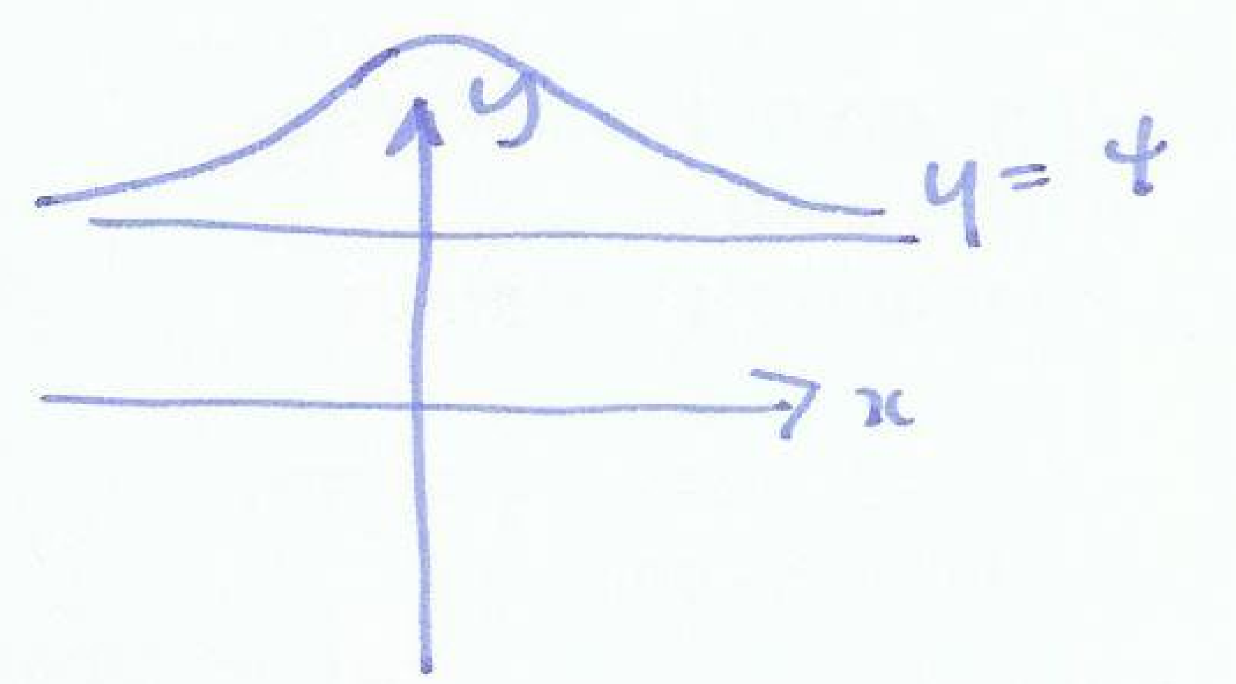
Q: what if $\deg(p) = \deg(q)$: might have horizontal asymptotes.

Example $\dfrac{4x^2-2}{x^2+1}$ for large x, expect $\sim \dfrac{4}{1}$
(ratio of leading coeffs).

divide:

$$\begin{array}{r|l} & 4 \\ \hline x^2+1 & 4x^2-2 \\ & 4x^2-4 \\ \hline & \quad 2 \end{array}$$

$$\frac{4x^2-2}{x^2+1} = 4 + \frac{2}{x^2+1}$$

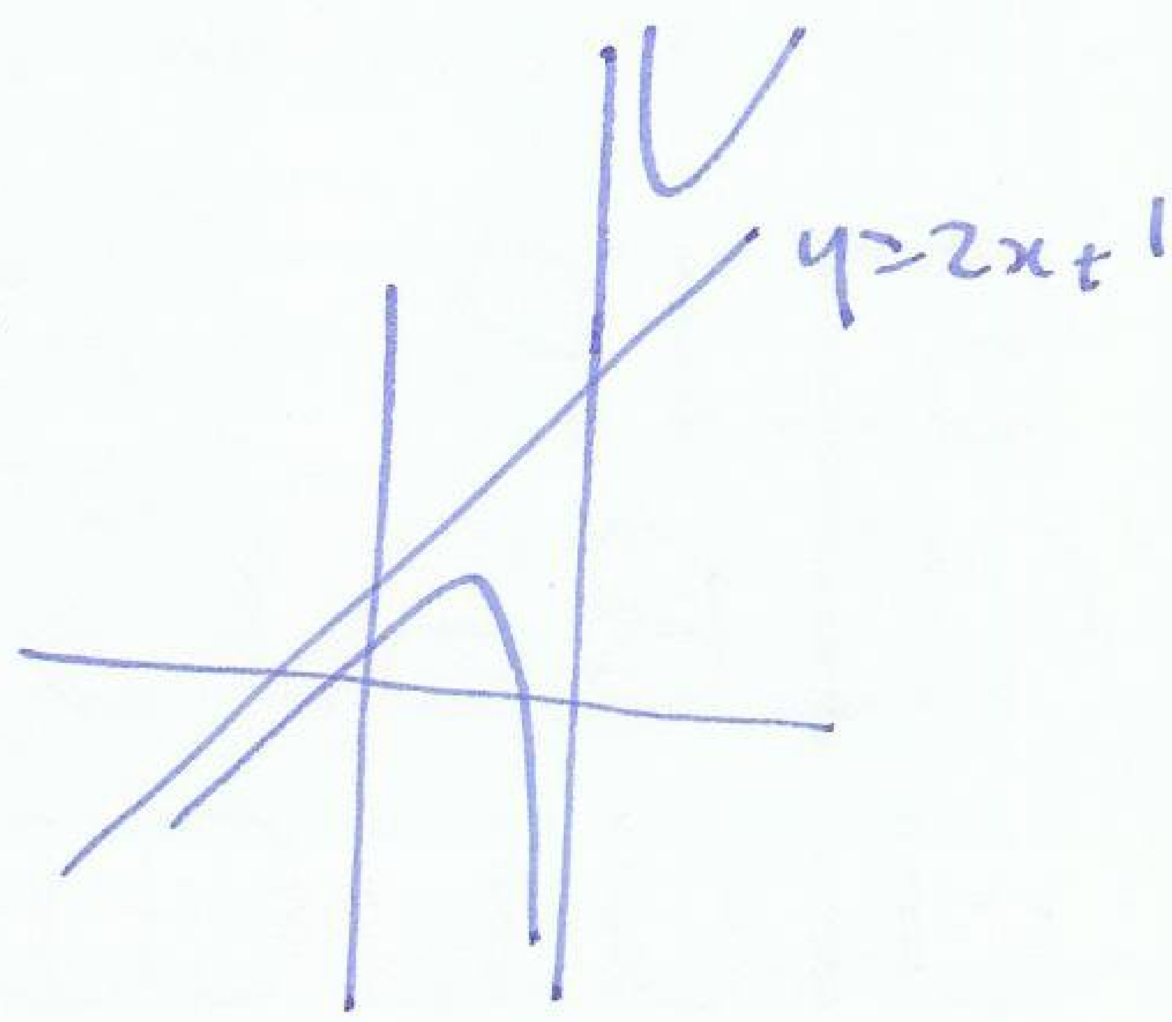


Diagonal asymptotes

example $\dfrac{2x^2-3x-1}{x-2} = 2x+1+\dfrac{1}{x-2}$

divide:

$$\begin{array}{r|l} & 2x+1 \\ \hline x-2 & 2x^2-3x-1 \\ & 2x^2-4x \\ \hline & \quad x-1 \\ & \quad x-2 \\ \hline & \quad\ \ 1. \end{array}$$

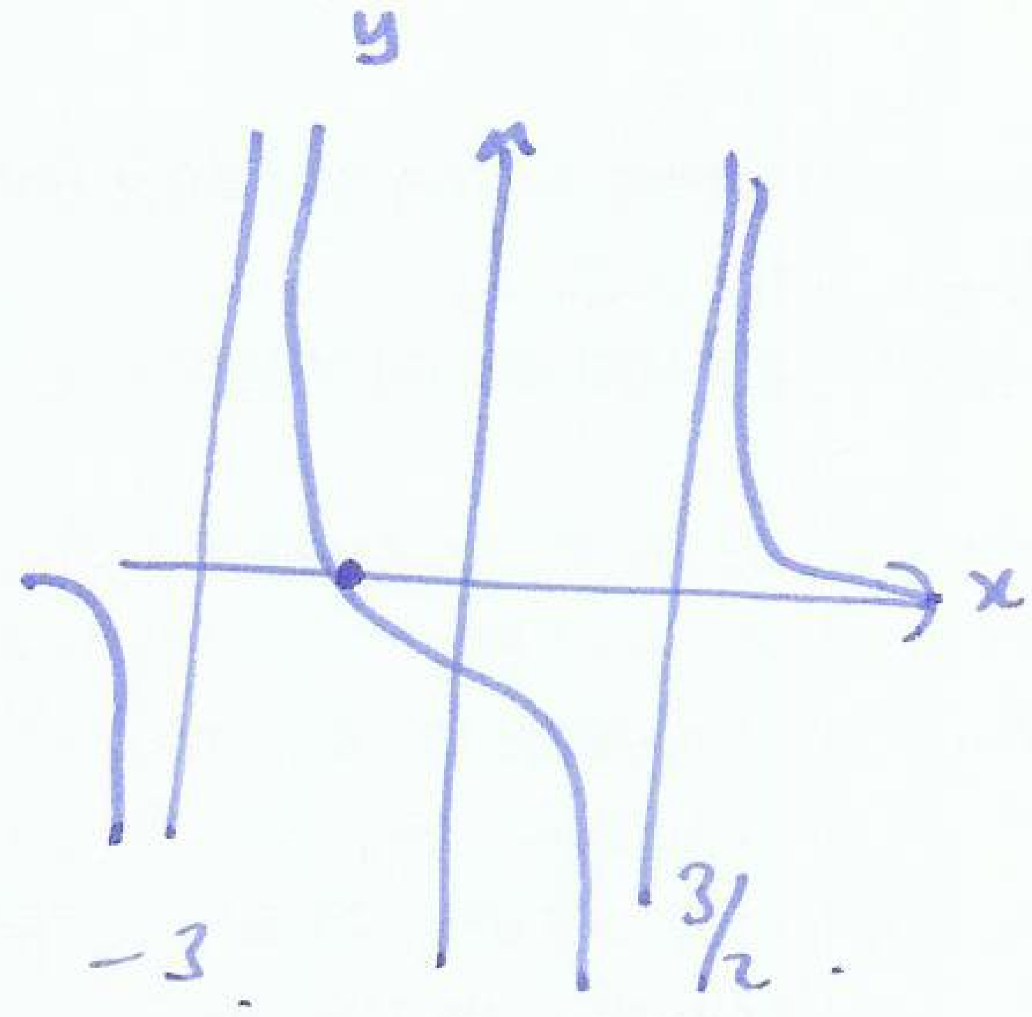


$x=2$.

graphing $\frac{p(x)}{q(x)}$: 1. find zeros in denominator.
(gives vertical asymptotes).

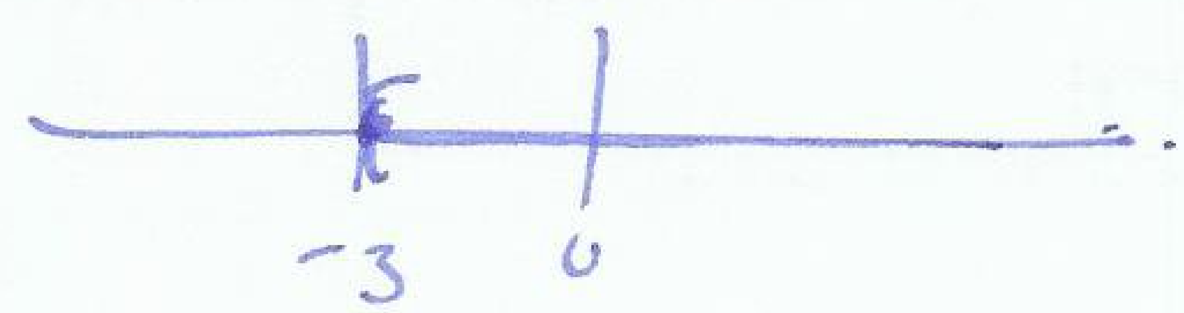
2. find horizontal / sloping asymptotes.
3. find zeros of function (zeros of $p(x)$).
4. find $f(0)$
5. check signs to find general shape.

<u>Example</u> $\frac{2x+3}{3x^2+7x-6} = \frac{2x+3}{(3x-2)(x+3)}$



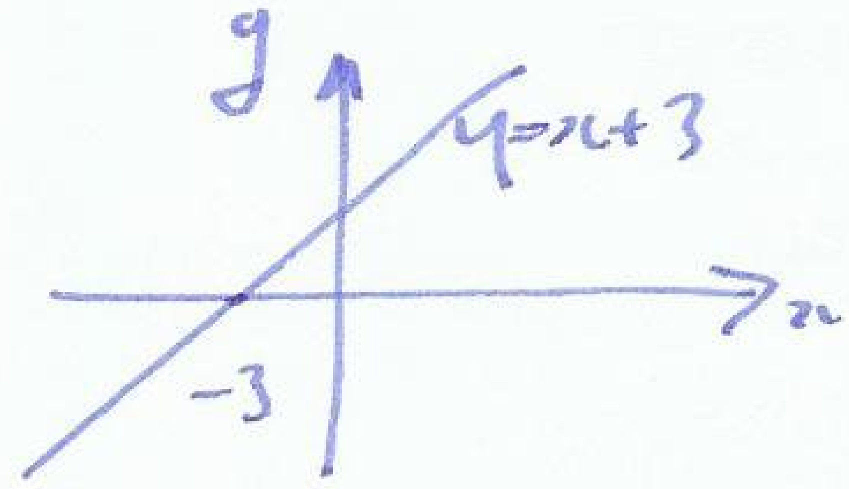
§4.6 Polynomial and rational inequalities

<u>inequalities</u> example $x > -3$.



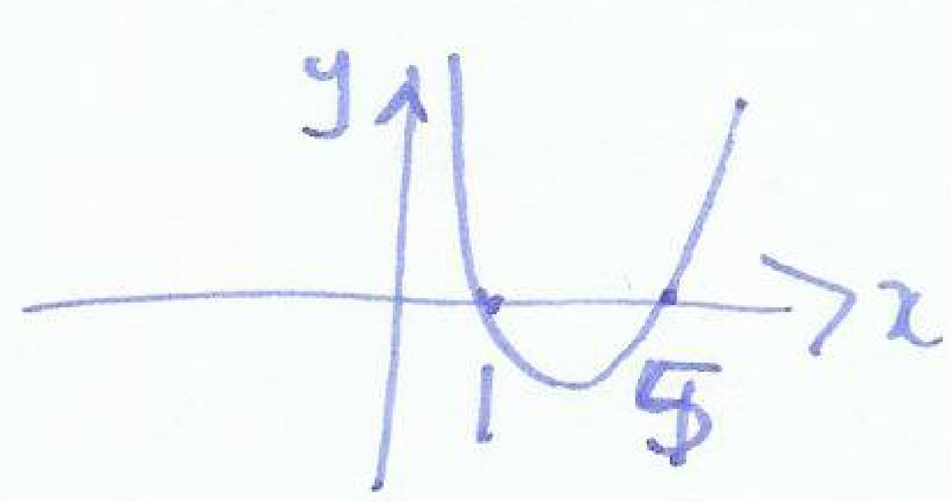
~~f(x)>~~ same as $x+3 > 0$.

same as $f(x) = x+3 > 0$



quadratic inequality: $x^2-4x-5 > 0$

draw $x^2-4x-5 = (x-5)(x-1)$



$x \geq 5$, ~~and~~ or $x \leq 1$.

$(-\infty, 1] \cup [5, \infty)$.

useful facts about polynomials $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

- if a polynomial has real coefficients then any complex roots come in pairs $a+bi, a-bi$.
- if a polynomial has rational coefficients then any rational roots $\frac{p}{q}$ have the property that $p \mid a_0$ $q \mid a_n$

where the polynomial is $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

Example $f(x) = 3x^4 - 11x^3 + 10x - 4 = (x+1)(x-\frac{2}{3})(3x^2-12x+6)$

Q: are there any rational roots $\frac{p}{q}$? $p \mid 4$ $q \mid 3$.

all possibilities: $p = \pm 1 \; \pm 2 \; \pm 4$

$q = \pm 1 \; \pm 3$.

$1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$.

Drawing rational functions $\frac{p(x)}{q(x)}$.

1. find zeros in denominator (gives vertical asymptotes).
2. find horizontal or sloping asymptotes.
3. find zeros of function (zeros of numerator)
4. find $f(0)$.
5. check signs to find general shape.

Synthetic division

$$
\begin{array}{r|l}
 & 4x^2 + 5x + 11 \\
\hline
x-2 & 4x^3 - 3x^2 + x + 7 \\
 & 4x^3 - 8x^2 \\
\hline
 & \quad 5x^2 + x \\
 & \quad 5x^2 - 10x \\
\hline
 & \qquad 11x + 7 \\
 & \qquad 11x - 22 \\
\hline
 & \qquad\qquad 29
\end{array}
$$

$$
\begin{array}{r|rrrr}
2 & 4 & -3 & 1 & 7 \\
 & & 8 & 10 & 22 \\
\hline
 & 4 & 5 & 11 & 29
\end{array}
$$