<u>Defn</u> A non-singular matrix A is <u>orthogonal</u> if $A^{-1} = A^T$.

equivalently : $A^T A = I_n$.

<u>Observation</u> if $\{\underline{v}_1, \dots \underline{v}_n\}$ is an orthonormal basis then

$A = [\underline{v}_1 \dots \underline{v}_n]$ is an orthogonal matrix.

<u>Thm</u> A is orthogonal iff the columns (rows) of A form an orthonormal basis. $\square$.

<u>Thm</u> If A is a symmetric matrix then A is diagonalizable, ~~and~~ $A = PDP^{-1}$ and we may choose P to be an orthogonal matrix.

<u>Proof</u> (maybe later : just need to show there are n eigenvectors).

<u>Thm</u> A is symmetric iff A is diagonalizable by an orthogonal matrix.

<u>Proof</u> $(PDP^{-1})^T = (P^{-1})^T D^T P^T = PDP^{-1}$ $\square$.

§9.3 Dynamical systems

linear differential equations $\underline{x}'(t) = A\underline{x}(t)$

if A diagonalizable $A = PDP^{-1}$ $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

then $\underline{x}'(t) = D\underline{x}(t)$ has solutions

$$x_1'(t) = \lambda_1 x_1 \qquad x_1(t) = c_1 e^{\lambda_1 t}$$

$$\vdots \qquad\qquad \vdots$$

$$x_n'(t) = \lambda_n x_n \qquad x_n(t) = c_n e^{\lambda_n t}.$$

so $\underline{x}'(t) = A\underline{x}(t)$ has solutions $P\begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t}\end{bmatrix}$.

check LHS: $P\begin{bmatrix} c_1\lambda_1 e^{\lambda_1 t} \\ \vdots \\ c_n\lambda_n e^{\lambda_n t}\end{bmatrix}$ RHS: $PDP^{-1}P\begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t}\end{bmatrix}$.

$$= PD\begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t}\end{bmatrix}$$

$$= P\begin{bmatrix} \lambda_1 c_1 e^{\lambda_1 t} \\ \vdots \\ \lambda_n c_n e^{\lambda_n t}\end{bmatrix} \quad \checkmark.$$

Q: what happens if A not diagonalizable?

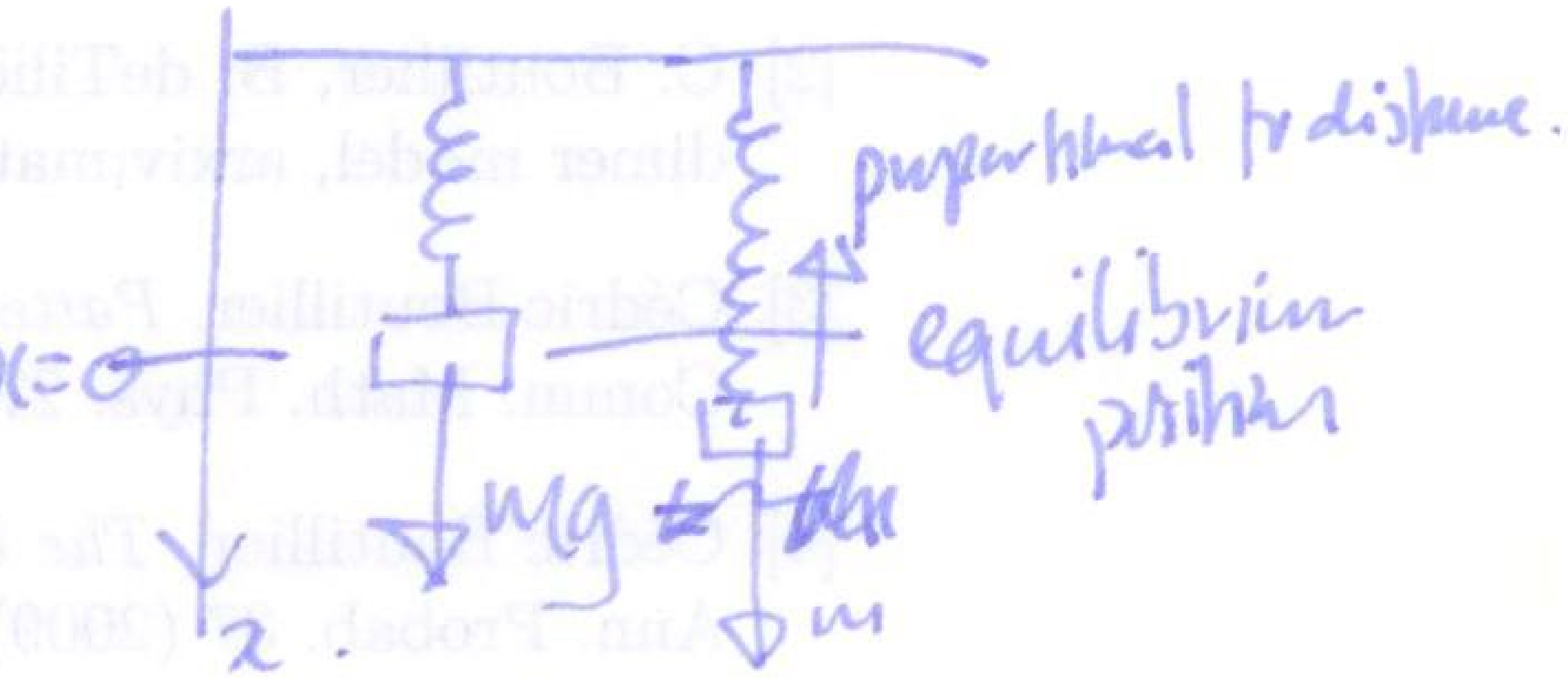
special case $n=2$ $\begin{bmatrix} x_1 \\ x_2\end{bmatrix}$ $\begin{bmatrix} x \\ y\end{bmatrix}$.

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

$\underline{x}'(t) = A\underline{x}(t)$ $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$

Example simple harmonic motion



$mx''(t) = -kx(t)$

$mx''(t) + kx(t) = 0$] 2nd order!

equivalent to:

set $y = y(t) = x'(t) = \frac{dx}{dt}$

then $y'(t) = x''(t)$ so we get $my'(t) + kx(t) = 0$

$$x'(t) = y(t).$$

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i.e. $x'(t) = 0x(t) + y(t)$

$y'(t) = -\frac{k}{m} x(t) + 0y(t)$

$\underline{x}'(t) = A\underline{x}(t) \quad A = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}.$

eigenvalues: $(-\lambda)(-\lambda) + \frac{k}{m} = 0 \quad \lambda^2 + \frac{k}{m} = 0 \quad \lambda = \pm i\sqrt{k/m}.$

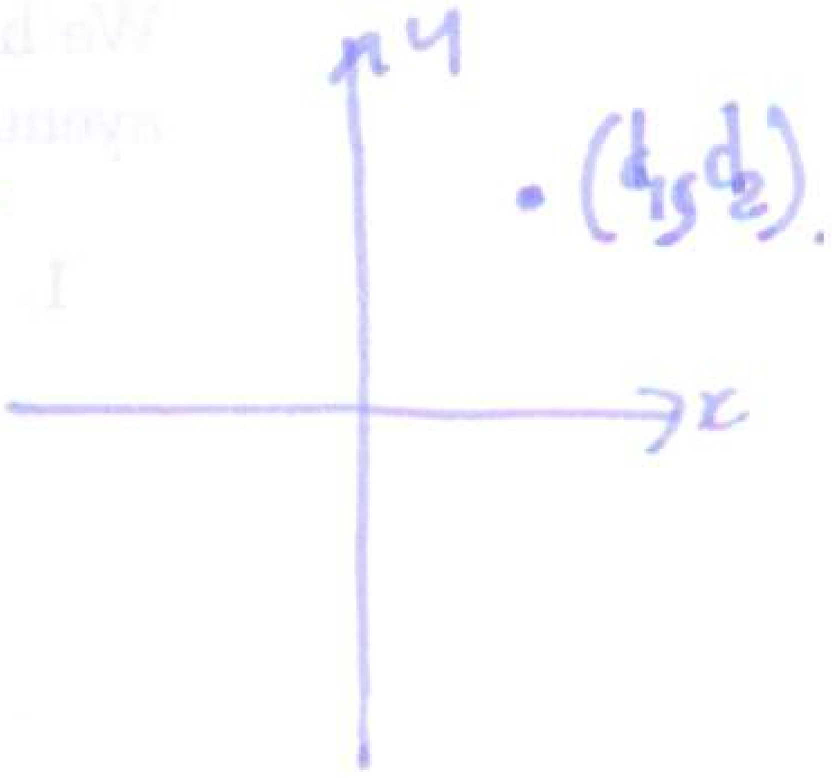
eigenvectors: $\underline{v}_1 = \begin{bmatrix} -i\sqrt{m/k} \\ 1 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} i\sqrt{m/k} \\ 1 \end{bmatrix} \quad \lambda_1 = + \quad \lambda_2 = -$

so solutions are $c_1 \underline{v}_1 e^{i\sqrt{m/k}\,t} + c_2 \underline{v}_2 e^{-i\sqrt{m/k}\,t}$

special case: $\sqrt{m/k} = 1.$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \lambda_1 = i \quad \lambda_2 = -i \quad \underline{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$

initial condition: $\underline{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cancel{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}} \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \quad \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix}.$



want: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cancel{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}} = c_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{i0} + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{i0}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -ic_1 + ic_2 \\ c_1 + c_2 \end{bmatrix} \quad \left[\begin{array}{cc|c} -i & i & 1 \\ 1 & 1 & 1 \end{array}\right] . \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -i & i & 1 \end{array}\right].$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2i & 1+i \end{array}\right] \quad c_2 = \frac{1+i}{2i} = \frac{-1+i}{-2} = \frac{1-i}{2}$

$c_1 = 1 - c_2 = 1 - \frac{1-i}{2} = \frac{+1+i}{2}.$

so solution is $+\frac{1+i}{2}\begin{bmatrix}-i\\1\end{bmatrix}e^{it} + \frac{1-i}{2}\begin{bmatrix}i\\1\end{bmatrix}e^{-it}$

$$\frac{1}{2}\begin{bmatrix}-ie^{it}+e^{it}+ie^{-it}+e^{-it}\\ +e^{it}+ie^{it}+e^{-it}-ie^{-it}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}(e^{it}+e^{-it}) + \frac{1}{2}i(e^{-it}-e^{it})\\ +\frac{1}{2}(e^{it}+e^{-it}) + \frac{1}{2}i(e^{it}-e^{-it})\end{bmatrix}$$

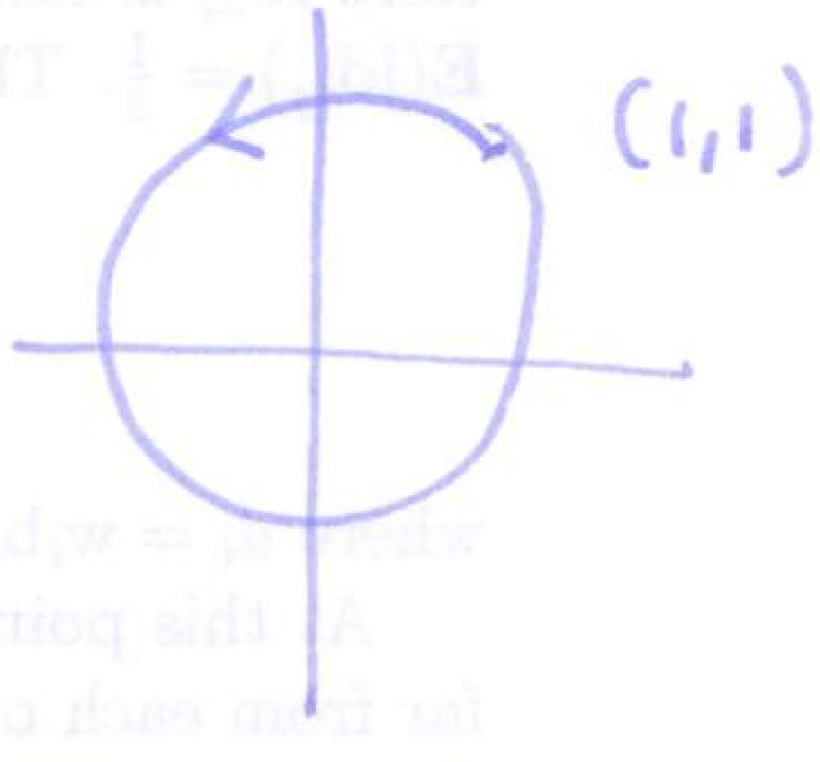
$e^{it} = \cos t + i\sin t$

$e^{-it} = \cos t - i\sin t$

$$= \begin{bmatrix}\cos t + \sin t\\ \cos t - \sin t\end{bmatrix}.$$

i.e. $x(t) = \cos t + \sin t$

$y(t) = x'(t) = \cos t - \sin t$



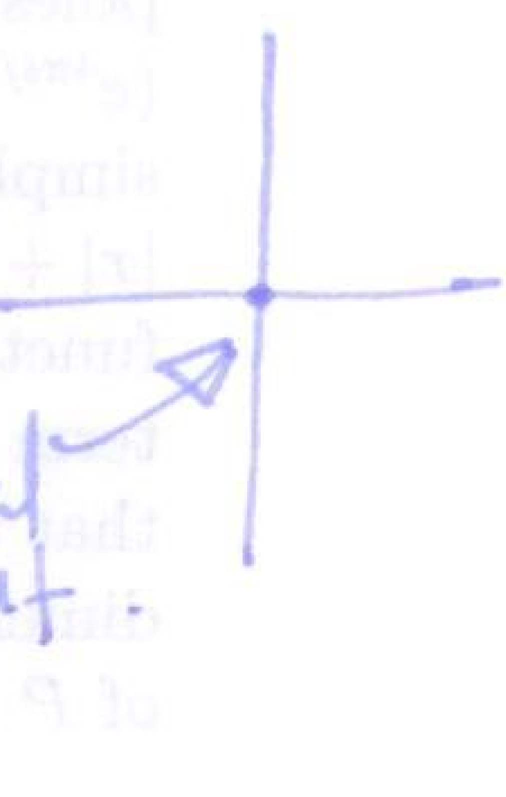
<u>Def^n</u> a point in the phase plane where $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, $\underline{x}'(t) = 0$

is called a <u>critical point</u> (fixed point) (equilibrium point)

$\underline{x}'(t) = A\underline{x}(t)$ when is $A\underline{x}(t) = \underline{0}$? when $\underline{x}(t) = \underline{0}$!

(assume A has full rank)

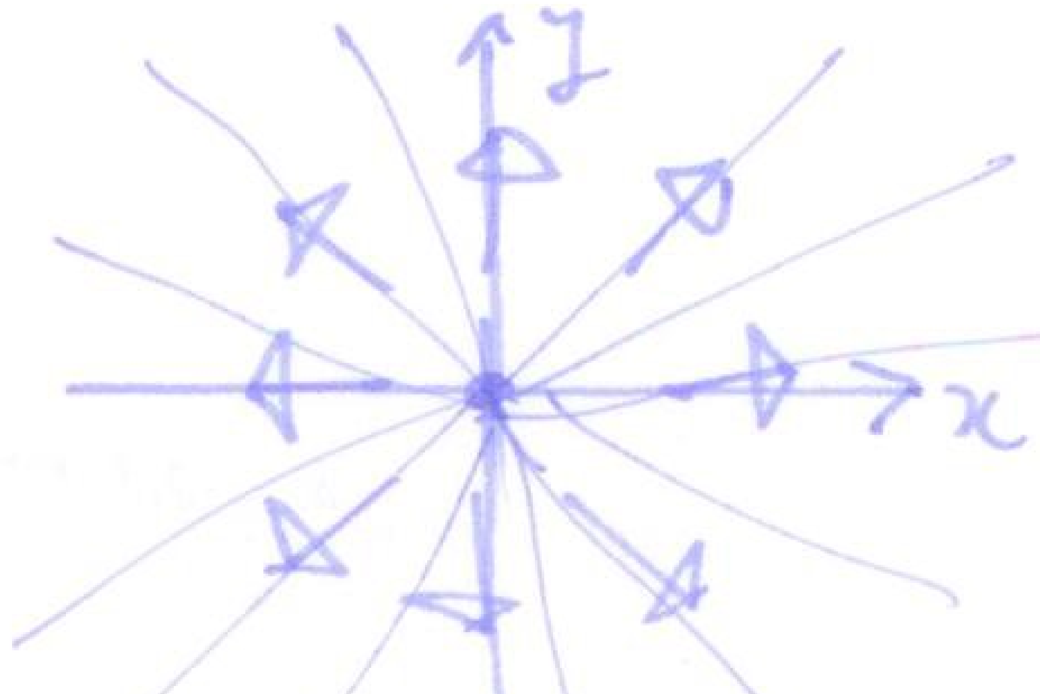
<u>Q</u>: what direction should you go in at



$\underline{x}(t) = \begin{bmatrix}c_1\\c_2\end{bmatrix}$? $\underline{A}$ $A\begin{bmatrix}c_1\\c_2\end{bmatrix}$

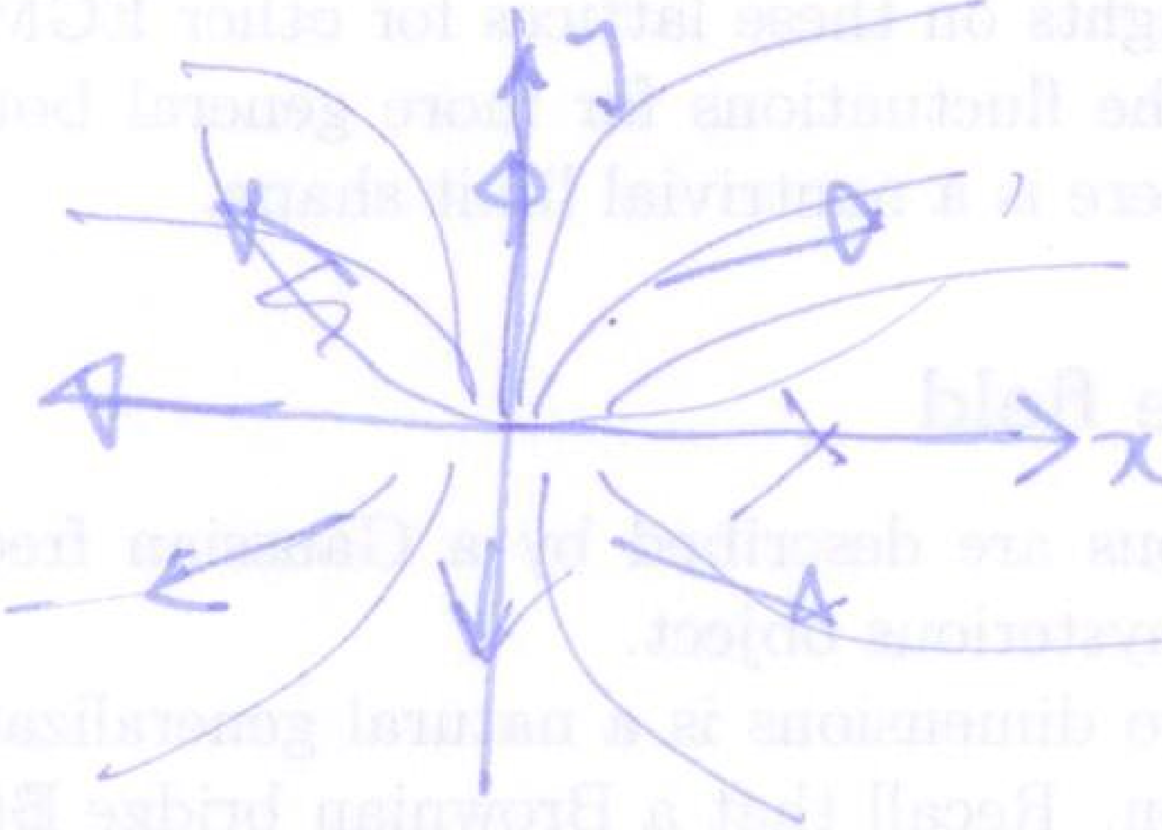
this gives a vector field.

Examples $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.



$\underline{x}(t) = \begin{bmatrix} c_1 e^t \\ c_2 e^t \end{bmatrix}$ unstable.

$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

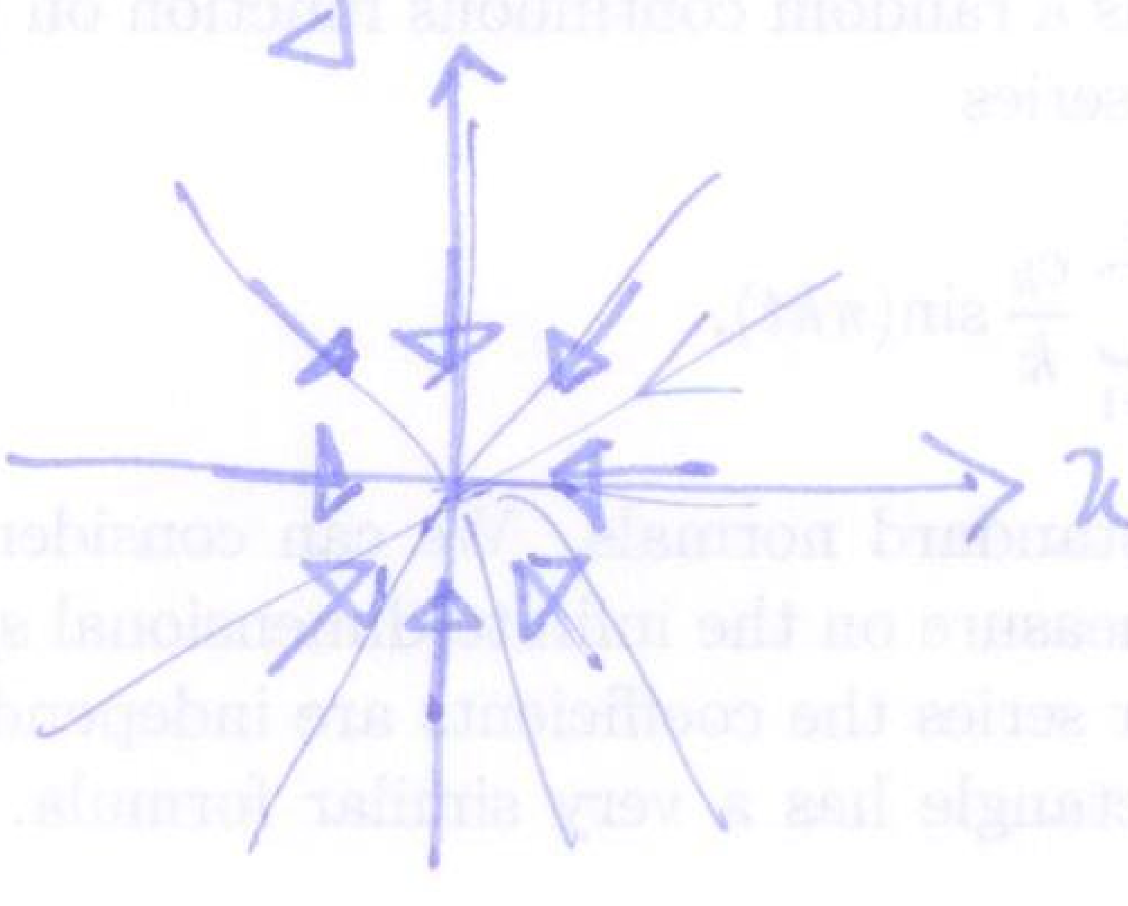


$\underline{x}(t) = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^t \end{bmatrix}$.

$\frac{x}{c_1} = e^{2t}$ $\frac{y}{c_2} = e^t$.

$\frac{x}{c_1} = \frac{y^2}{c_2^2}$

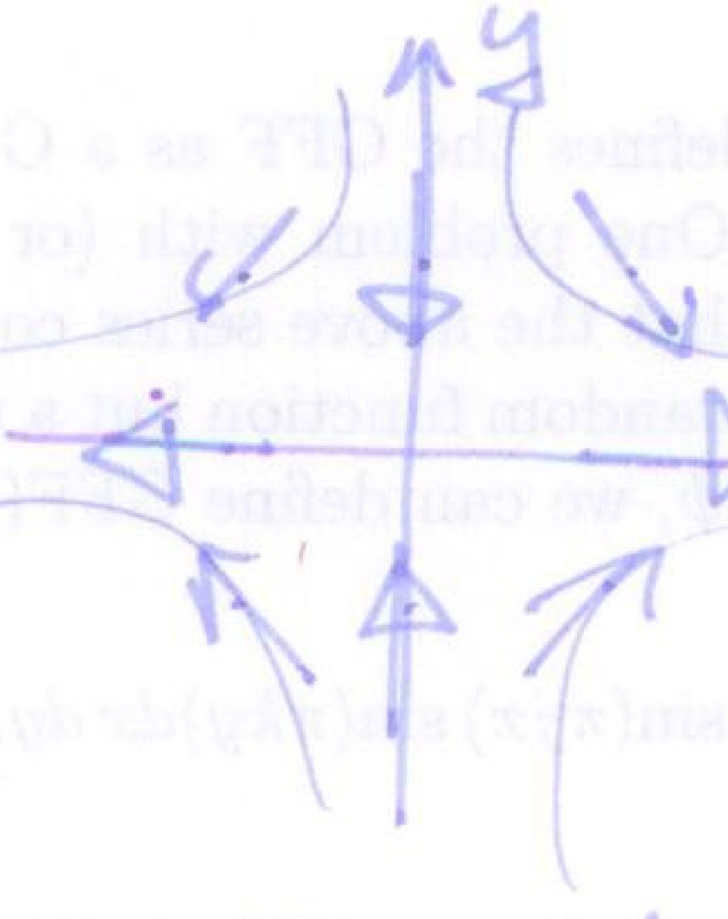
$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$



$\underline{x}(t) = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-t} \end{bmatrix}$.

stable .

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



$\underline{x}(t) = \begin{bmatrix} c_1 e^t \\ c_2 e^{-t} \end{bmatrix}$

$\frac{x}{c_1} = e^t = \frac{c_2}{y}$

$xy = c_2/c_1$.

unstable saddle point

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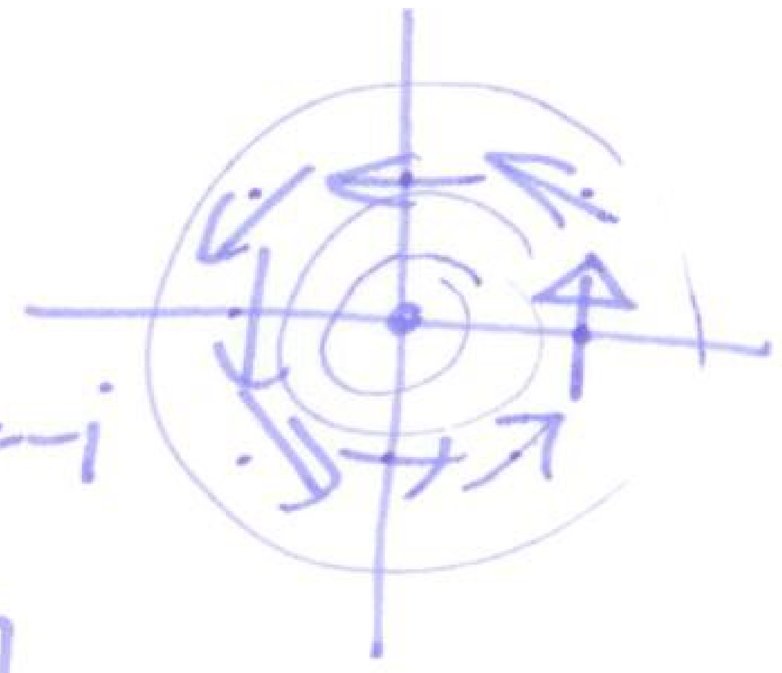
$A = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$ ~~eig~~

eigenvalues : $\lambda^2 + 1 = 0$

$\lambda = \pm i \quad \lambda_1 = i \quad \lambda_2 = -i$



eigenvectors : $\underline{v}_1 \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \quad \underline{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\underline{v}_2 \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

solutions : $c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{it} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-it} = \begin{bmatrix} c_1 e^{it} + c_2 e^{-it} \\ -ic_1 e^{it} + ic_2 e^{-it} \end{bmatrix}$.

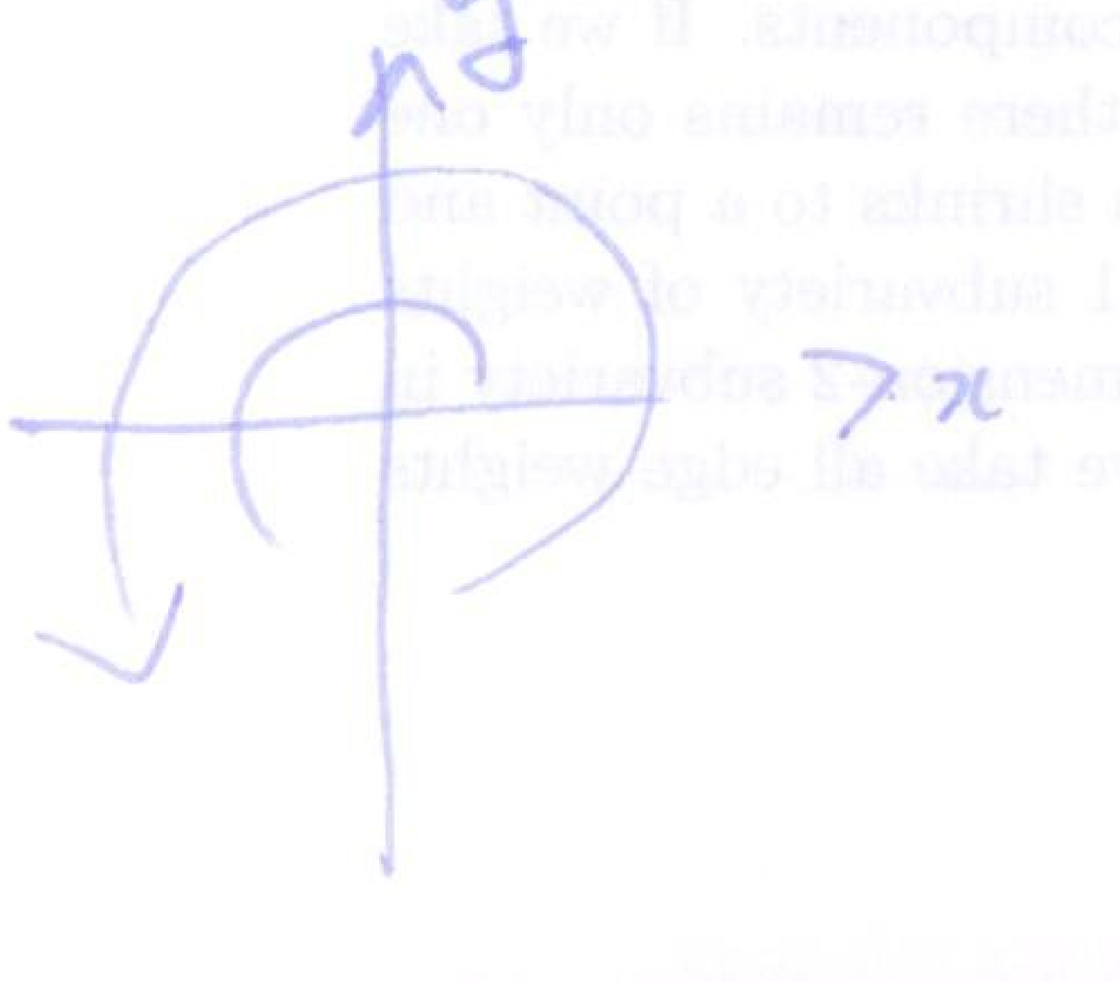
$e^{it} = \cos t + i \sin t \qquad \begin{bmatrix} (c_1 + c_2)\cos t + i(c_1 - c_2)\sin t \\ i(c_2 - c_1)\cos t + (c_1 + c_2)\sin t \end{bmatrix}$

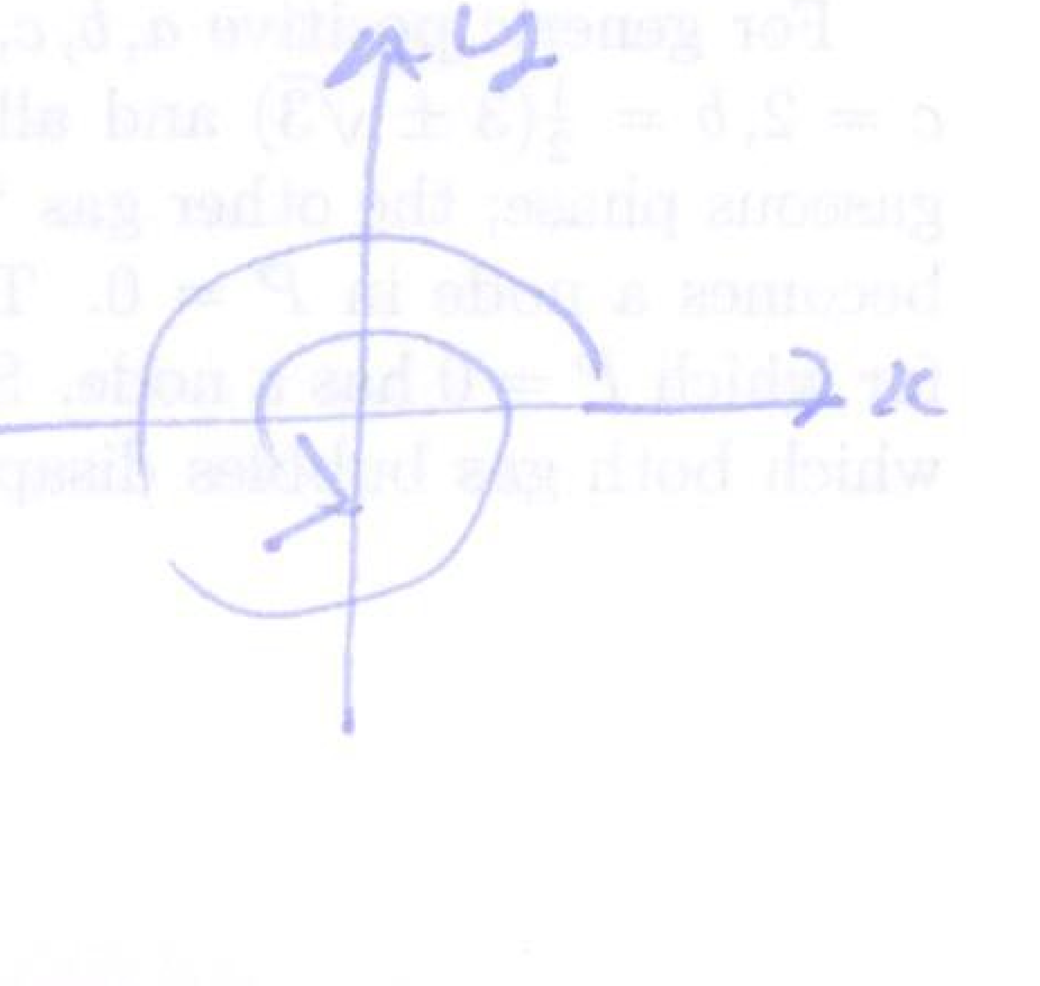
general case of complex eigenvalues : $a \pm bi$

sol^ns $\quad e^{a \pm bi} = e^a e^{bi} = e^a(\cos bt + i \sin bt)$.

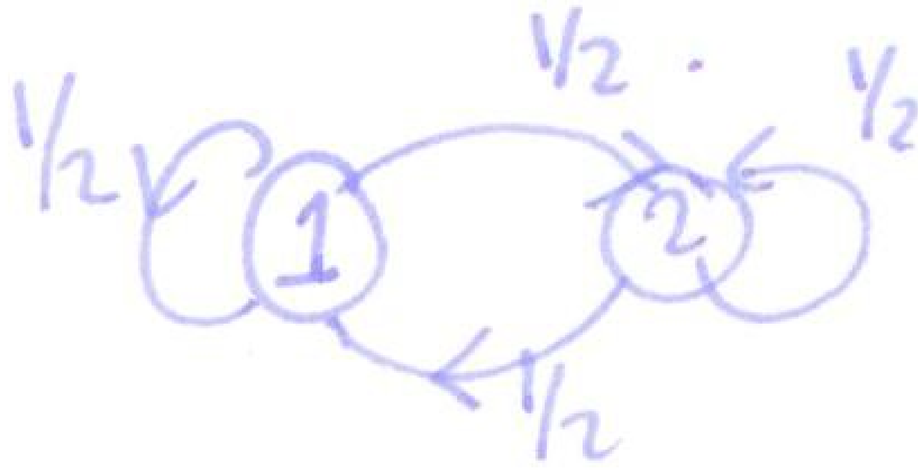
so stability depends on whether $a > 0 \quad a < 0$

$a > 0$ $\qquad\qquad$ $a < 0$





§2.5 Markov chains



$t=0$: in state 1. $(1,0)$.

$t=1$: prob in 1 $\frac{1}{2}$. $(\frac{1}{2},\frac{1}{2})$.

2 $\frac{1}{2}$.

spose at $t=0$ in state 1 w prob p $(p+q=1)$ (p,q).

2 q

$t=1$ in state 1 w prob $\frac{1}{2}p+\frac{1}{2}q$

2 $\frac{1}{2}p+\frac{1}{2}q$

i.e. if distribution at time t is $\begin{bmatrix} p \\ q \end{bmatrix}$

$t+1$ is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$

transition matrix T.

invariant/equilibrium distribution is $P\underline{v} = \underline{v}$

time	
$t=0$	$\underline{x}$
$t=1$	$P\underline{x}$
$t=2$	$P^2\underline{x}$
$\vdots$	
$t=k$	$P^k\underline{x}$ ← what does this look like?