§8.2 Diagonalization

recall $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\lambda_1, \underline{v_1}$ $\lambda_2 \underline{v_2}$ $L(\underline{x}) = A\underline{x}$.

suppose we have a vector in terms of $\{\underline{v_1}, \underline{v_2}\}$.

i.e. $\underline{v} = c_1 \underline{v_1} + c_2 \underline{v_2}$

$$L(\underline{v}) = L(c_1 \underline{v_1} + c_2 \underline{v_2}) = c_1 L(\underline{v_1}) + c_2 L(\underline{v_2})$$

$$= c_1 \lambda_1 \underline{v_1} + c_2 \lambda_2 \underline{v_2}.$$

Q: what is matrix for L in terms of $\{\underline{v_1}, \underline{v_2}\}$.

A $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$! $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{bmatrix}$.

Let P be the change of basis matrix $P_{\cdot \leftarrow S} = [\underline{v_1} \ \underline{v_2}]$

then $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} P$

Def^n A matrix B is <u>similar</u> or <u>conjugate</u> to a matrix A if there is a non-singular matrix P s.t. $B = P^{-1} A P$.

(symmetric, transitive)

Def^n A is <u>diagonalizable</u> if A is conjugate to a diagonal matrix.

Th^m similar matrices have the same eigenvalues. (in fact same char polys).

Proof suppose $B = P^{-1} A P$

$$\chi_B = \det(B - \lambda I) = \det(P^{-1} A P - \lambda I)$$

$$= \det(P^{-1} A P - \lambda P^{-1} I P)$$

$$= \det\left(P^{-1}(A-\lambda I)P\right)$$

$$= \det(P^{-1})\det(A-\lambda I)\det(P) = \det(A-\lambda I) = \chi_A \quad \square.$$

<u>Observation</u> if D is diagonal, then the eigenvalues of D are the entries on the main diagonal.

$$D = \begin{bmatrix} d_1 & 0 & & 0 \\ 0 & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \quad \text{then } \det(D-\lambda I) = \det\begin{pmatrix} d_1-\lambda & & 0 \\ & d_2-\lambda & \\ 0 & & \ddots \; d_n-\lambda \end{pmatrix}$$

$$= (d_1-\lambda)\dots(d_n-\lambda).$$

<u>Thm</u> A is diagonalizable iff it has n linearly independent eigenvectors.

<u>Proof</u> • Suppose A is diagonalizable, i.e. $\exists$ non-singular matrix P s.t. ~~$A = D = P^{-1}AP$. $A = 8$~~. $D = P^{-1}AP$.

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ eigenvalues with eigenvector $\underline{e_j}$ standard basis vectors.

<u>claim</u> $P^{\cancel{-1}}\underline{e_j}$ is an eigenvector of $A = PDP^{-1}$

$$AP\underline{e_j} = PDP^{-1}P\underline{e_j} = PD\underline{e_j} = P\lambda_j\underline{e_j} = \lambda_j P\underline{e_j}$$

so $P\underline{e_j}$ has eigenvalue λ_j.

$\underline{e_j}$ basis so P non-singular $\Rightarrow$ $P\underline{e_j}$ a basis.

• Suppose A has n linearly independent eigenvectors $\underline{v_1}, \dots \underline{v_n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$. set $P = \left[\underline{v_1} \dots \underline{v_n}\right]$.

consider $P^{-1}AP\underline{e_j} = P^{-1}A\underline{v_j} = P^{-1}\lambda\underline{v_j} = \lambda P^{-1}\underline{v_j} = \lambda\underline{e_j}$

so $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ diagonal. $\square$.

recall $\begin{bmatrix}2&1\\1&1\end{bmatrix}$ diagonalizable $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ not diagonalizable.

Thm Eigenvectors corresponding to distinct eigenvalues are linearly indep.

Proof A matrix, let $\underline{v_1} \dots \underline{v_k}$ be eigenvectors with eigenvalues $\lambda_1 \dots \lambda_k$ all different. Suppose $c_1\underline{v_1} + c_2\underline{v_2} + \dots + c_k\underline{v_k} = \underline{0}$

then $$\underline{v_1} = \cancel{-\frac{c_2}{c_1}\underline{v_1}} - \frac{c_2}{c_1}\underline{v_2} - \dots - \frac{c_k}{c_1}\underline{v_k}. \quad (*)$$

$$A\underline{v_1} = \cancel{-\frac{c_2}{c_1}A\underline{v_1}} - \frac{c_2}{c_1}A\underline{v_2} \dots - \frac{c_k}{c_1}A\underline{v_k}.$$

$$\lambda_1 \underline{v_1} = \cancel{-\frac{c_2}{c_1}} - \frac{c_2}{c_1}\lambda_2 \underline{v_2} \dots - \frac{c_k}{c_1}\lambda_k \underline{v_k}$$

$\lambda_1(*)$ $$\lambda_1 \underline{v_1} = -\frac{c_2}{c_1}\lambda_1 \underline{v_2} \dots - \frac{c_k}{c_1}\lambda_1 \underline{v_k}$$

so $$\underline{0} = -\frac{c_2}{c_1}(\underbrace{\lambda_2-\lambda_1}_{\neq 0})\underline{v_2} - \dots - \frac{c_k}{c_1}(\underbrace{\lambda_k-\lambda_1}_{\neq 0})\underline{v_k}$$

so $\{\underline{v_2} \dots \underline{v_k}\}$ are dependent. continue until you have just two vectors $c_1\underline{v_1} + c_2\underline{v_2} = 0 \Rightarrow \underline{v_1}, \underline{v_2}$ parallel but then

$A\underline{v_1} = \lambda_1\underline{v_1}$ $\quad \underline{v_1} = d_1\underline{v_2}$ $\quad A\underline{v_1} = \lambda_1\underline{v_1} = d_1\lambda_1\underline{v_2}$ $\not\#$. $\square$

$A\underline{v_2} = \lambda_2\underline{v_2}$ $\qquad = d_1\lambda_2\underline{v_2}$

Corollary If the eigenvalues are all distinct then the matrix is diagonalizable.

Observation ① if A, B are similar/conjugate then they have the same eigenvalues, ~~and you can~~

$A = P^{-1}BP$ suppose $\underline{v}, \lambda$ are eigenvector/eigenvalue for B i.e. $B\underline{v} = \lambda\underline{v}$ then $P^{-1}\underline{v}, \lambda$ is an eigenvector/value for A

$$P^{-1}BPP^{-1}\underline{v} = P^{-1}B\underline{v} = P^{-1}\lambda\underline{v} = \lambda P^{-1}\underline{v}.$$

② $(P^{-1}DP)^2 = P^{-1}DPP^{-1}DP = P^{-1}D^2P$. $D = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$ $D^2 = \begin{bmatrix}\lambda_1^2 & 0\\ 0 & \lambda_2^2\end{bmatrix}$.

Application : differential equations.

linear differential equations in two vars: $\frac{dx}{dt} = ax+by$

$\underline{x} = \begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$ $\qquad \frac{dy}{dt} = cx+dy$

or $\frac{d\underline{x}}{dt} = A\underline{x}$ ⊛ $\quad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$. $\qquad D = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$.

how can we solve this? suppose $A = P^{-1}DP$ D diagonal.

solve $\dot{\underline{x}} = D\underline{x}$ $\qquad \frac{dx}{dt} = \lambda_1 x \qquad x = c_1 e^{\lambda_1 t}$

$\qquad \frac{dy}{dt} = \lambda_2 y \qquad y = c_2 e^{\lambda_2 t}$

i.e. $\underline{x}(t) = \begin{bmatrix}c_1 e^{\lambda_1 t}\\ c_2 e^{\lambda_2 t}\end{bmatrix}$. then solution to ⊛ is $P^{-1}\underline{x}(t)$.

$\frac{dx}{dt} = P^{-1}DP\underline{x}$ plug in $\frac{d}{dt}(P^{-1}\underline{x}(t)) = P^{-1}\frac{d}{dt}\underline{x}(t) = \begin{bmatrix}\lambda_1 c_1 e^{\lambda_1 t}\\ \lambda_2 c_2 e^{\lambda_2 t}\end{bmatrix}$.

$P^{-1}DPP^{-1}\underline{x} = P^{-1}D\underline{x} = P^{-1}\begin{bmatrix}\lambda_1 c_1 e^{\lambda_1 t}\\ \lambda_2 c_1 e^{\lambda_2 t}\end{bmatrix}$. $\square$.