§10.2 Kernel and range

V, W vector spaces kernel of L $\ker(L) = \{ v \in V \mid L(\underline{v}) = \underline{0}_W \}$.
$L: V \to W$ linear map

Thm $\ker(L)$ is a subspace of V.

Thm $L: V \to W$ is injective iff $\ker L = \{\underline{0}_V\}$

Proof $\Rightarrow$ ✓.

$\Leftarrow$ suppose $\ker L = \{\underline{0}_V\}$. ~~suppose~~

and $L(\underline{u}) = L(\underline{v})$ then $L(\underline{u}) - L(\underline{v}) = \underline{0}_W$.

$$L(\underline{u}) - L(\underline{v}) = L(\underline{u} - \underline{v})$$

$\Rightarrow \underline{u} - \underline{v} = \underline{0}_V \Rightarrow \underline{u} = \underline{v}$. $\square$.

Corollary if $L(\underline{x}) = \underline{b}$ and $L(\underline{y}) = \underline{b}$, then $\underline{x} - \underline{y} \in \ker L$.

Defn $\text{image}(L) = \{ L(\underline{x}) \mid \underline{x} \in V \}$ (book calls this the range)

if $\text{image}(L) = W$ then L is onto.

Thm ~~If~~ $\text{image}(L)$ is a subspace of W.

Proof suppose $\underline{w}_1, \underline{w}_2 \in$ image of W, then $\exists$ $\underline{v}_1, \underline{v}_2$ s.t.

$L(\underline{v}_1) = \underline{w}_1$, $L(\underline{v}_2) = \underline{w}_2$ so $L(\underline{v}_1 + \underline{v}_2) = L(\underline{v}_1) + L(\underline{v}_2) = \underline{w}_1 + \underline{w}_2$.

$L(k\underline{v}_1) = k L(\underline{v}_1) = k\underline{w}_1$. $\square$

Thm (Rank nullity) $\dim(\ker L) + \dim(\text{image } L) = \underbrace{\dim V}_{n}$. (82)

Proof Let $k = \dim(\ker L)$

if $k = n$, then $\ker L = V$ so $n + 0 = n$ ✓.

so assume $k < n$.

choose a basis for $\ker L$ $\{\underline{v}_1, \dots \underline{v}_k\} = S$.

extend this to a basis for V $\{\underline{v}_1, \dots \underline{v}_k, \underline{v}_{k+1} \dots \underline{v}_n\}$.

set $T = \{L(\underline{v}_{k+1}), \dots, L(\underline{v}_n)\}$

claim T is a basis for image(L).

span: suppose $\underline{w} \in$ image(L) then $\exists\, \underline{v}$ s.t. $L(\underline{v}) = \underline{w}$.

write $\underline{v}$ in terms of basis $\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k + c_{k+1}\underline{v}_{k+1} + \dots + c_n\underline{v}_n$

then $L(\underline{v}) = \underbrace{c_1 L(\underline{v}_1) + c_2 L(\underline{v}_2) + \dots + c_k L(\underline{v}_k)}_{0} + c_{k+1} L(\underline{v}_{k+1}) + \dots + c_n L(\underline{v}_n)$.

so span(T) contains image(L).

independent: suppose $c_{k+1} L(\underline{v}_{k+1}) + \dots + c_n L(\underline{v}_n) = \underline{0}_W$

then $L(c_{k+1}\underline{v}_{k+1} + \dots + c_n\underline{v}_n) = \underline{0}_W$

$\Rightarrow$ $c_{k+1}\underline{v}_{k+1} + \dots + c_n\underline{v}_n \in \ker L$

so can write $c_{k+1}\underline{v}_{k+1} + \dots + c_n\underline{v}_n = c_1\underline{v}_1 + \dots + c_k\underline{v}_k$

$c_1\underline{v}_1 + \dots + c_k\underline{v}_k - c_{k+1}\underline{v}_{k+1} - \dots - c_n\underline{v}_n = \underline{0}_V \Rightarrow c_i = 0$ for all i. □

($k=0$ basis for V then $L(\underline{v}_i)$ basis for image as 1-1)

Corollary $L: V \to W$ linear map $\dim V = \dim W$

then if L 1-1 then onto

if L onto then 1-1. $\square$.

§10.3 The matrix of a linear transformation

Thm Let $L: V \to W$ be a linear map $\dim V = n$, $\dim W = m$

Let $S = \{\underline{v}_1, \dots \underline{v}_n\}$ be a basis of V

$T = \{\underline{w}_1, \dots \underline{w}_m\}$ be a basis of W.

Then there is a unique matrix A s.t. $[L(\underline{x})]_T = A[\underline{x}]_S$

Proof Let A be the matrix with j-th column given by

$[L(\underline{v}_j)]_T$ $\quad A = [L(\underline{v}_1)_T \, L(\underline{v}_2)_T \dots L(\underline{v}_n)_T]$

then if $\underline{x} = c_1\underline{v}_1 + \dots + c_n\underline{v}_n$ then

$$A[\underline{x}]_S = A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [L(\underline{v}_1)_T \dots L(\underline{v}_n)_T]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= c_1 L(v_1)_T + \dots + c_n L(v_n)_T.$$

$$= L(c_1\underline{v}_1 + \dots + c_n\underline{v}_n)_T$$

$$= [L(\underline{x})]_T \quad \text{as required.}$$

uniqueness: 1st col of A given by $A\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = L(\underline{v}_1)_T$

where $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_T = c_1\underline{v}_1 + 0 \dots + 0$ so no choice in cols. $\square$.

<u>Example</u> $L:\mathbb{R}^3 \to \mathbb{R}^2$ $\quad L\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+y\\y-z\end{bmatrix}$.

L wrt standard bases: $A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & -1\end{bmatrix}$.

L wrt $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ $\quad \underline{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

$T = \{\underline{w}_1, \underline{w}_2\}$

$\underline{w}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ $\underline{w}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

find L wrt S, T.

$L(\underline{v}_1) = \begin{bmatrix}1\\-1\end{bmatrix}$ $\quad L(\underline{v}_2) = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad L(\underline{v}_3) = \begin{bmatrix}2\\0\end{bmatrix}$.

need to write these in terms of T.

i.e. $c_1\underline{w}_1 + c_2\underline{w}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

solve. $\left[\underline{w}_1\ \underline{w}_2 \,\middle|\, L(\underline{v}_1) \,\middle|\, L(\underline{v}_2) \,\middle|\, L(\underline{v}_3)\right]$

$$\left[\begin{array}{cc|ccc}1 & -1 & 1 & 1 & 2\\2 & 1 & -1 & 0 & 0\end{array}\right] \rightsquigarrow \underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{I_n}\underbrace{\left|\begin{array}{c|c|c}0 & 1/3 & 2/3\\-1 & -2/3 & -4/3\end{array}\right]}_{A}.$$

$L: V \to W$ linear

$\uparrow$ basis S $\quad \uparrow$ basis T

change of basis changes the matrix!

$A_{S,T}$.

new basis S' $\quad T'$

there is a change of basis map $P_{S' \leftarrow S} : V \longrightarrow V$ $\quad P_{T' \leftarrow T} : W \longrightarrow W$.

$[\underline{v}]_S \mapsto [\underline{v}]_{S'}$ $\quad [\underline{w}]_T \mapsto [\underline{w}]_{T'}$

new matrix $A_{S',T'}$ $[\underline{v}]_{S'} \longmapsto [L(\underline{v})]_{T'}$

$$\begin{tikzcd}
{[v]_S} \arrow[r, mapsto, "A_{S,T}"] \arrow[d, leftrightarrow, "P_{S'\leftarrow S}"] & {[L(\underline{v})]_T} \arrow[d, "P_{T'\leftarrow T}"] \\
{[v]_{S'}} \arrow[r, mapsto, "A'"'] & {[L(\underline{v})]_{T'}}
\end{tikzcd}$$

so $A' = A_{S',T'} = P_{T'\leftarrow T}\, A_{S,T}\, P_{S'\leftarrow S}^{-1}$

<u>special case</u> $L: V_S \to V_S$ linear map from V to itself.

use same basis in V! $A_{S,S}$.

change basis $A' = PAP^{-1}$ only one choice of matrix!

<u>§8.1 Eigenvectors,</u>

warning: we'll need complex numbers/vectors/matrices.

<u>Def^n</u> A $n \times n$ matrix. a number λ is an <u>eigenvalue</u> of A if there is a non-zero vector $\underline{x}$ such that $A\underline{x} = \lambda\underline{x}$

the non-zero vector is called an <u>eigenvector</u> with eigenvalue λ.

<u>Example</u> I_n $I_n\underline{x} = \underline{x}$ every non-zero vector is an eigenvector with eigenvalue 1.

Example $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $1/2$.

note $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ not an eigenvector.

How to find eigenvectors/values.

want $A\underline{x} = \lambda \underline{x}$

$A\underline{x} - \lambda I_n \underline{x} = \underline{0}$

$(A - \lambda I_n)\underline{x} = \underline{0}$ Q when does this have non-trivial sols?

A when $\det(A - \lambda I_n) = 0$. $\leftarrow$ degree n poly in λ. (known)

Example $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$

$= (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1$ $\lambda = \dfrac{3 \pm \sqrt{9-4}}{+2}$

$\lambda = \dfrac{3 \pm \sqrt{5}}{2}$ (possible) eigenvalues. $\lambda_1 = \dfrac{3+\sqrt{5}}{2}$ $\lambda_2 = \dfrac{3-\sqrt{5}}{2}$.

now solve $(A - \lambda_1 I)\underline{x} = 0$.

$$\begin{bmatrix} 2 - \frac{3+\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{3+\sqrt{5}}{2} \end{bmatrix} \quad \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \quad \lambda_1 \ \underline{v_1} = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix}.$$

check!

$(A - \lambda_2 I)\underline{x} = \underline{0}$

$$\begin{bmatrix} 2 - \frac{3-\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{3-\sqrt{5}}{2} \end{bmatrix} \quad \begin{bmatrix} \frac{-1-\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix}. \quad \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

<u>Def^n</u> A $n \times n$ matrix. $\det(A-\lambda I_n)$ is the <u>characteristic polynomial</u> of A. $\det(A-\lambda I_n)=0$ is the <u>characteristic equation</u>

<u>Th^m</u> A $n\times n$ matrix is singular iff 0 is an eigenvalue of A.

<u>Proof</u> $\det(A-0I_n)=0 \Leftrightarrow \det(A)=0$ $\square$.

<u>Th^m</u> The eigenvalues of A are precisely the roots of the characteristic equation.

<u>Proof</u> suppose $A\underline{x}=\lambda\underline{x}$, then $(A-I\lambda)\underline{x}=\underline{0}$ has a non-trivial solution, so ~~λ is a ro~~ $A-I\lambda$ is singular, so $\det(A-I\lambda)=0$.
i.e. λ is a root of char poly

suppose λ satisfies $\det(A-\lambda I)=0$. Then $A-\lambda I$ is singular
so $(A-\lambda I)x=\underline{0}$ has a non-trivial solution. $\underline{v}$

so $(A-\lambda I)\underline{v}=0 \Rightarrow A\underline{v}=\lambda\underline{v}$ so λ is an eigenvalue $\square$.

<u>Example</u> $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ $\det(A-\lambda I)=(1-\lambda)(1-\lambda)-0=0$

$(1-\lambda)^2=0$
i.e. one eigenvalue $+1$ with multiplicity 2.

<u>Q:</u> how many eigenvectors? $A-I=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}. \Rightarrow y=0.$

so (up to multiples) only eigenvalue of is $\begin{bmatrix}1\\0\end{bmatrix}$