$= \underline{v}\cdot\underline{w}_i - (\underline{v}\cdot\underline{w}_1)\underline{w}_1\cdot\underline{w}_i - \cdots (\underline{v}\cdot\underline{w}_i)\underline{w}_i\cdot\underline{w}_i - \cdots - (\underline{v}\cdot\underline{w}_m)\underline{w}_m\cdot\underline{w}_i$

$= \underline{v}\cdot\underline{w}_i - \underline{v}\cdot\underline{w}_i = \underline{0}$ as required □.

so $V = W + W^\perp$

but $W \cap W^\perp = \{\underline{0}\}$ so $V = W \oplus W^\perp$. □

do example.

Thm W subspace of $\mathbb{R}^n$, then $(W^\perp)^\perp = W$

Proof $w \in W$, the w orthogonal to every vector in $W^\perp$ so $w \in (W^\perp)^\perp$, so $W \subset (W^\perp)^\perp$.

converse: suppose $\underline{v} \in (W^\perp)^\perp$, any $\underline{v}$ can be written a $\underline{v} = \underline{w} + \underline{u}$ with $\underline{w} \in W$, $\underline{u} \in W^\perp$

consider $\underbrace{\underline{u}\cdot\underline{v}}_{=0} = \underbrace{\underline{u}\cdot\underline{w}}_{=0} + \underline{u}\cdot\underline{u} \Rightarrow \|\underline{u}\|^2 = 0 \Rightarrow \underline{u} = 0$.

so $\underline{v} \in W$. Therefore $W = (W^\perp)^\perp$. □.

Spaces associated to a matrix A $(m\times n)$ $A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

null space / kernel of A $(\subset \mathbb{R}^n)$ $A^T (n\times m): \mathbb{R}^n \longrightarrow \mathbb{R}^m$

row space of A $(\subset \mathbb{R}^n)$

null space / kernel of A^T $(\subset \mathbb{R}^m)$.

column space of A (image). $(\subset \mathbb{R}^m)$

$A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

kernel A | image A.

row space | kernel A^T.

$\underline{\text{Thm}}$ A $(m\times n)$ matrix

a) null space / kernel A orthogonal complement of row space of A.

b) null space / kernel A^T orthogonal complement of col space / image of A

$\underline{\text{Proof}}$ check dimensions: spose $\text{rank}(A) = r$

then $\text{nullity}(A) = n - r$ (rank-nullity theorem).

	space	dim	
so	$\ker(A)$	$n-r$	
	row space	r	
	$\ker(A^T)$	$m-r$	(rank nullity for A^T).
	col space	r	(dim row space = dim col space).

$x \in \ker(A) \iff A\underline{x} = 0$ let A have rows $\underline{v}_1, \dots, \underline{v}_m$ $\begin{bmatrix} \underline{v}_1 \\ \vdots \\ \underline{v}_m \end{bmatrix} = A$

so $A\underline{x} = \begin{bmatrix} \underline{v}_1 \\ \vdots \\ \underline{v}_m \end{bmatrix} \underline{x} = \begin{bmatrix} \underline{v}_1 . \underline{x} \\ \vdots \\ \underline{v}_m . \underline{x} \end{bmatrix} = \underline{0}$ $(\underline{x} \in \ker A)$

so $\underline{x} . \underline{v}_i = 0$ $\underline{x}$ orthogonal to $\underline{v}_i$ for all $\underline{v}_i$

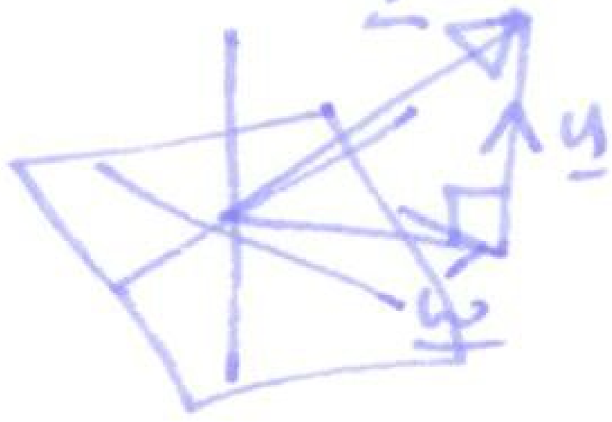
$\underline{v}_i$ span row space, so $\underline{x}$ orthogonal to row space, for all $\underline{x} \in \ker A$.

so $\ker A \perp$ row space. $\dim(\ker A) + \dim(\text{row space}) = \dim(\mathbb{R}^n) = n$

so $\mathbb{R}^n = \ker A \oplus$ row space.

for $\ker A^T$, col space, ~~same~~ just replace A with A^T. $\square$

Projections $W < V$



any vector $\underline{v} \in V$ can be written, uniquely as $\underline{v} = \underline{w} + \underline{u}$

for $\underline{w} \in W$, $\underline{u} \in W^{\perp}$

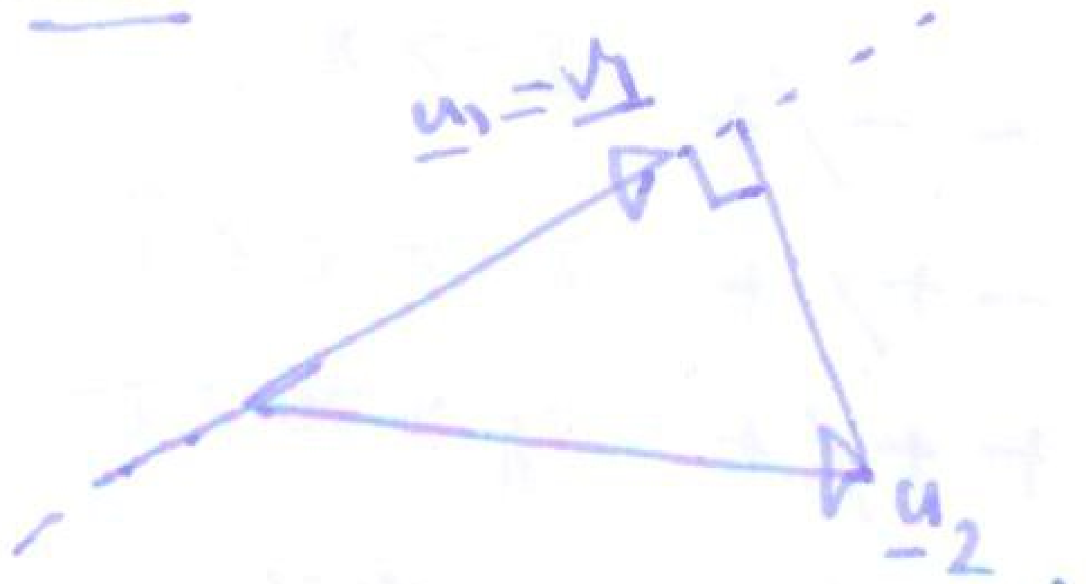
we say $\underline{w}$ is the orthogonal projection of $\underline{v}$ onto W $\mathrm{proj}_W(\underline{v})$

so $\mathrm{proj}_W : V \longrightarrow W$.

if $\{\underline{w}_1, \dots, \underline{w}_k\}$ is an orthonormal basis for W, then

$$\underline{w} = (\underline{v}\cdot\underline{w}_1)\underline{w}_1 + (\underline{v}\cdot\underline{w}_2)\underline{w}_2 + \cdots + (\underline{v}\cdot\underline{w}_k)\underline{w}_k .$$

Recall Gram–Schmidt : corresponds to subtracting projection at each step..

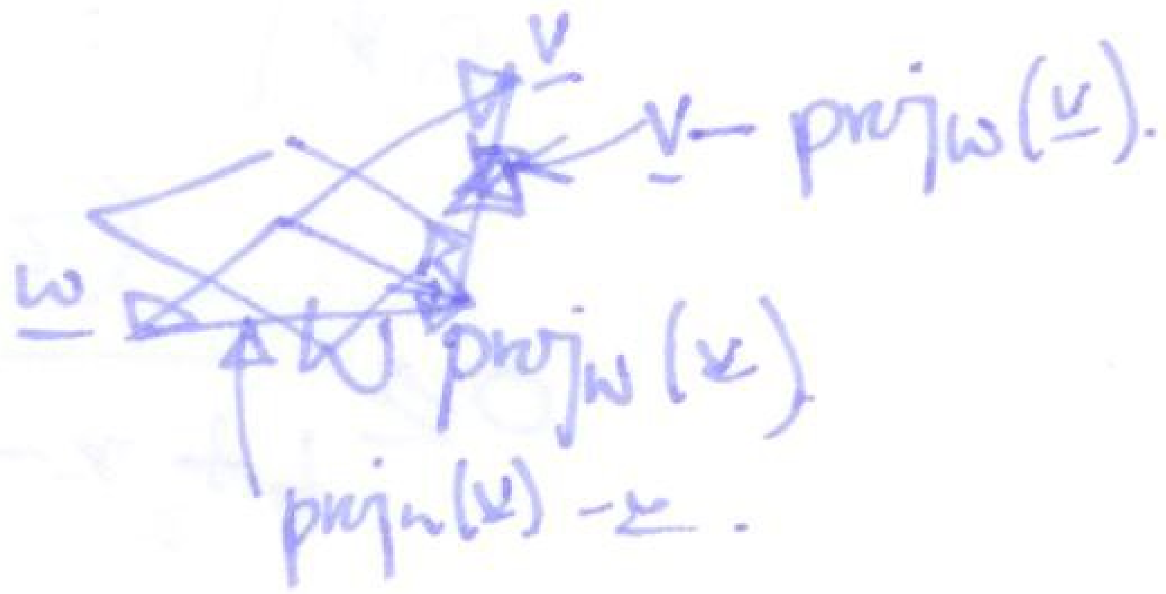


$$\underline{v}_2 = \underline{u}_2 - \mathrm{proj}_{\mathrm{span}(\underline{v}_1)} \underline{u}_2 \quad . \quad \text{etc} \dots$$

Useful fact Thm $W < \mathbb{R}^n$. Then for any vector $\underline{v} \in \mathbb{R}^n$, the closest vector to $\underline{v}$ in W is $\mathrm{proj}_W(\underline{v})$.

i.e. $\|\underline{v} - \underline{w}\|$ is minimized by $\underline{w} = \mathrm{proj}_W(\underline{v})$.

Proof $\underline{v} \in V$, consider any vector $\underline{w} \in W$.



$$\underline{v} - \underline{w} = \underbrace{\left(\underline{v} - \mathrm{proj}_W(\underline{v})\right)}_{\in W^{\perp}} + \underbrace{\left(\mathrm{proj}_W \underline{v} - \underline{w}\right)}_{\in W.}$$

$$\|\underline{v} - \underline{w}\|^2 = (\underline{v} - \underline{w})\cdot(\underline{v} - \underline{w})$$

$$= \Big[\underbrace{\big(\underline{v} - \mathrm{proj}_W(\underline{v})\big)}_{\in W} + \underbrace{\big(\mathrm{proj}_W \underline{v} - \underline{w}\big)}_{\in W^\perp}\Big] \cdot \Big[\underbrace{\big(\underline{v} - \mathrm{proj}_W(\underline{v})\big)}_{\in W} + \underbrace{\big(\mathrm{proj}_W \underline{v} - \underline{w}\big)}_{\in W^\perp}\Big].$$

$$= \underbrace{\|\underline{v} - \mathrm{proj}_W(\underline{v})\|^2}_{\text{doesn't depend on } \underline{w}!} + \underbrace{\|\mathrm{proj}_W \underline{v} - \underline{w}\|^2}_{\text{minimized uniquely by choosing } \underline{w} = \mathrm{proj}_W \underline{v}} \quad \square.$$

§10.1 Linear transformations

Def: V, W vector spaces. a linear map $L : V \to W$ is a map such that

a) $L(\underline{u} + \underline{v}) = L(\underline{u}) + L(\underline{v})$ for all $\underline{u}, \underline{v} \in V$

b) $L(k\underline{u}) = kL(\underline{u})$. for all $k \in \mathbb{R}, \underline{v} \in V$.

Thm/Observations 1) $L(c_1\underline{v_1} + c_2\underline{v_2} + \cdots + c_k\underline{v_k}) = c_1 L(\underline{v_1}) + \cdots + c_k L(\underline{v_k})$.

2) $L(0_{\underline{V}}) = 0_{\underline{W}}$

3) $L(\underline{u} - \underline{v}) = L(\underline{u}) - L(\underline{v})$.

Proof 2): $0_{\underline{V}} + 0_{\underline{V}} = 0_{\underline{V}}$ so $L(0_{\underline{V}}) + L(0_{\underline{V}}) = L(0_{\underline{V}})$.

$$L(0_{\underline{V}}) = 0_{\underline{W}}.$$

Corollary if $L(0_{\underline{V}}) \neq 0_{\underline{W}}$ then L is not linear.

Thm let $L : V \to W$ be a linear map, and let $\{\underline{v_1}, \ldots \underline{v_n}\}$ be a basis of V. Then L is determined by $\{L(\underline{v_1}), \ldots, L(\underline{v_n})\}$.

Proof $\underline{u} \in V$, then $\underline{u} = c_1\underline{v_1} + \cdots + c_n\underline{v_n}$.

so $L(\underline{u}) = c_1 L(\underline{v_1}) + \cdots + c_n L(\underline{v_n})$. $\square$.