§6.9 Orthogonal complements

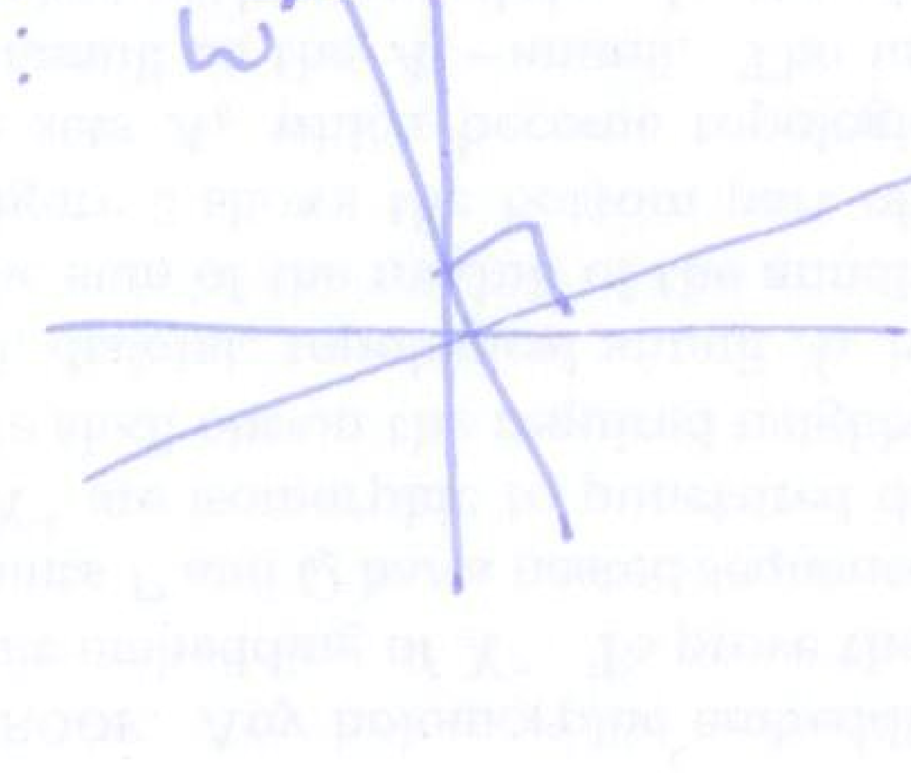
V vector space with inner product

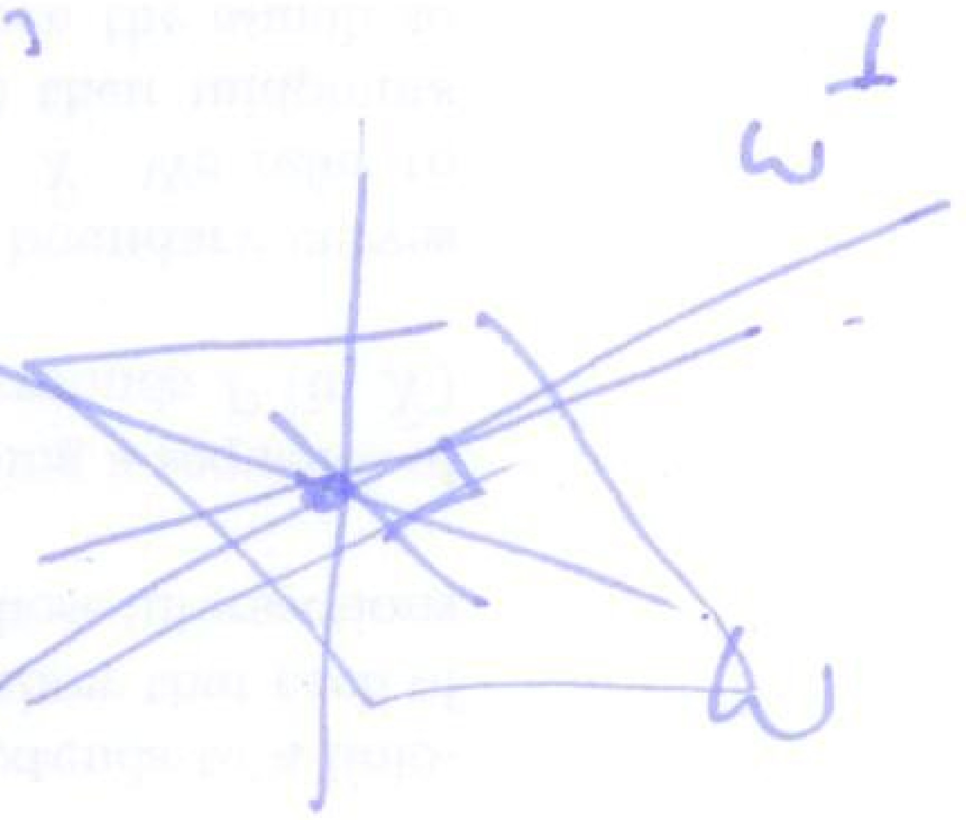
W subspace of V

Defn $W^\perp$ orthogonal complement of W

$$W^\perp = \{ \underline{v} \in V \mid \underline{v}.\underline{w} = 0 \text{ for all } \underline{w} \in W \}.$$

Example $\mathbb{R}^2$:





Thm W subspace, then

a) $W^\perp$ is a subspace

b) $W \cap W^\perp = \{\underline{0}\}$

Proof a) check closure properties.

suppose $\underline{w}_1, \underline{w}_2 \in W^\perp$ so $\underline{w}_1.\underline{v} = 0$ for all $\underline{v} \in V$

$\underline{w}_2.\underline{v} = 0$ " "

addition:

then $(\underline{w}_1 + \underline{w}_2).\underline{v} = \underline{w}_1.\underline{v} + \underline{w}_2.\underline{v} = 0$ for all $\underline{v} \in V$

scalar mult: $(\lambda \underline{w}_1).\underline{v} = \lambda(\underline{w}_1.\underline{v}) = 0$ for all $\underline{v} \in V$.

b) suppose $\underline{u} \in W \cap W^\perp$ then $\underline{u}.\underline{u} = 0 \Rightarrow \underline{u} = \underline{0}$.

<u>Example</u> $W \subset \mathbb{R}^4$ with basis $\{\underline{w_1}, \underline{w_2}\}$ $\quad \underline{w_1} = (1,1,0,1)$

$\underline{w_2} = (0,-1,1,1)$

find a basis for $W^\perp$

$\underline{x} = (x_1, x_2, x_3, x_4) \in W^\perp$ then $\underline{x}\cdot\underline{w_1} = 0 \quad \underline{x}\cdot\underline{w_2} = 0$

pair of linear equations:

$$x_1 + x_2 + x_3 = 0$$
$$-x_2 + x_3 + x_4 = 0$$

i.e. solve

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 \end{array}\right] \qquad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & \underbrace{-1}_{s} & \underbrace{-1}_{t} & 0 \end{array}\right]$$

$x_1 = -s-2t$

$x_2 = s+t$

$x_3 = s$

$x_4 = t$

$$\underline{x} = s\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

so $\{(-1,1,1,0), (-2,1,0,1)\}$ is a basis for $W^\perp$.

<u>sums/ Direct sums</u>

Let W_1, W_2 be subspaces of V. Then $W_1 + W_2$ is the set of all vectors.

$\{\underline{w_1} + \underline{w_2} \mid \underline{w_1} \in W_1, \underline{w_2} \in W_2\}$.

<u>Exercise</u> check $W_1 + W_2$ is a subspace of V

Let W_1, W_2 be subspaces of V such that $W_1 + W_2 = V$

$W_1 \cap W_2 = \{\underline{0}\}$

<u>Exercise</u>: every vector $\underline{v}$ in V can be written as $\underline{v} = \underline{w_1} + \underline{w_2}$ for $\underline{w_1} \in W_1$ and $\underline{w_2} \in W_2$ in a <u>unique</u> way. We say $V = W_1 \oplus W_2$.

Proof $W_1 + W_2 = V$ so every vector $\underline{v} \in V$ can be written as

$\underline{v} = \underline{w_1} + \underline{w_2}$ for some $\underline{w_1} \in W_1$ $\underline{w_2} \in W_2$.

suppose $\underline{v} = \underline{w_1} + \underline{w_2}$ $\quad \underline{w_1}', \underline{w_1} \in W_1$

$\underline{v} = \underline{w_1}' + \underline{w_2}'$ $\quad \underline{w_2}, \underline{w_2}' \in W_2$.

then $$\underline{v} - \underline{v} = \underbrace{(\underline{w_1} - \underline{w_1}')}_{\in W_1} + \underbrace{(\underline{w_2} - \underline{w_2}')}_{\in W_2} = \underline{0}$$

so $$\underbrace{\underline{w_1} - \underline{w_1}'}_{\in W_1} = -\underbrace{(\underline{w_2} - \underline{w_2}')}_{\in \underline{W_2}}$$ but $W_1 \cap W_2 = \{\underline{0}\}$

so $\underline{w_1} - \underline{w_1}' = \underline{0}$

$\underline{w_2} - \underline{w_2}' = \underline{0}$ D.

Thm Let W be a subspace of ~~R~~ V finite dim(n).

then ~~$\mathbb{R}$~~ $V = W \oplus W^\perp$

Proof suppose $\dim W = m$, then ~~find~~ has basis with m elements.

Gram-Schmidt: find orthonormal basis $\{\underline{w_1}, \dots \underline{w_m}\}$ for W.

let $\underline{v} \in V$ set $\underline{w} = (\underline{v} \cdot \underline{w_1})\underline{w_1} + (\underline{v} \cdot \underline{w_2})\underline{w_2} + \dots + (\underline{v} \cdot \underline{w_m})\underline{w_m}$

then $\underline{w} \in W$. set $\underline{u} = \underline{v} - \underline{w}$.

then $\underline{v} = \underbrace{\underline{w}}_{\in W} + \underline{u}$ claim $\underline{u} \in W^\perp$

suffices to show $\underline{u} \cdot \underline{w_i} = 0$ for all basis vectors $\underline{w_i} \in W$.

$$\underline{u} \cdot \underline{w_i} = (\underline{v} - \underline{w}) \cdot \underline{w_i}$$

$$= \underline{v} \cdot \underline{w_i} - \left((\underline{v} \cdot \underline{w_1})\underline{w_1} + (\underline{v} \cdot \underline{w_2})\underline{w_2} + \dots + (\underline{v} \cdot \underline{w_m})\underline{w_m}\right) \underline{w_i}$$

$$= \underline{v}\cdot\underline{w}_i - (\underline{v}\cdot\underline{w}_1)\underline{w}_1\cdot\underline{w}_i - \cdots (\underline{v}\cdot\underline{w}_i)\underline{w}_i\cdot\underline{w}_i - \cdots - (\underline{v}\cdot\underline{w}_m)\underline{w}_m\cdot\underline{w}_i$$

$$= \underline{v}\cdot\underline{w}_i - \underline{v}\cdot\underline{w}_i = \underline{0}$$ as required □.

so $V = W + W^\perp$

but $W \cap W^\perp = \{\underline{0}\}$ so $V = W \oplus W^\perp$. □

<u>Thm</u> W subspace of $\mathbb{R}^n$, then $(W^\perp)^\perp = W$

<u>Proof</u> $w \in W$, the w orthogonal to every vector in $W^\perp$ so $w \in (W^\perp)^\perp$, so $W \subset (W^\perp)^\perp$.

converse: suppose $\underline{v} \in (W^\perp)^\perp$, any $\underline{v}$ can be written a

$\underline{v} = \underline{w} + \underline{u}$ with $\underline{w} \in W$, $\underline{u} \in W^\perp$

consider $\underbrace{\underline{u}\cdot\underline{v}}_{=0} = \underbrace{\underline{u}\cdot\underline{w}}_{=0} + \underline{u}\cdot\underline{u} \Rightarrow \|\underline{u}\|^2 = 0 \Rightarrow \underline{u} = 0.$

so $\underline{v} \in W$. Therefore $W = (W^\perp)^\perp$. □.

Spaces associated to a matrix A $(m \times n)$ $\quad A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

null space / kernel of A $(\subset \mathbb{R}^n)$ $\quad A^T (n \times m) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

row space of A $(\subset \mathbb{R}^n)$

null space / kernel of A^T $(\subset \mathbb{R}^m)$.

column space of A (image). $(\subset \mathbb{R}^m)$

$A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

kernel A $\quad$ image A.

row space $\quad$ kernel A^T.