§6.7 Coordinates and change of basis

Coordinates we will now consider ordered basis.

so $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is different from $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

note if $S = \{\underline{v}_1, \ldots \underline{v}_n\}$ is an (ordered) basis for V, then

every $\underline{v} \in V$ can be written uniquely as $\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v}_n$.

we will write $[\underline{v}]_S = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and call this the coordinate vector for $\underline{v}$ in the basis S.

order matters!

example $\mathbb{R}^2$: $S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ $\quad S_2 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

then $\underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then $[\underline{v}]_{S_1} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

but $[\underline{v}]_{S_2} = (-1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example $S = \{\underline{v}_1, \ldots, \underline{v}_4\}$ basis for $\mathbb{R}^4$

$\underline{v}_1 = [1,1,0,0]$ $\quad \underline{v}_2 = [2,0,1,0]$ $\quad \underline{v}_3 = [0,1,2,-1]$ $\quad \underline{v}_4 = [0,1,-1,0]$.

if $\underline{v} = [1,2,-6,2]$ find $[\underline{v}]_S$.

need to find $c_1, \ldots, c_4$ s.t. $\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + c_4\underline{v}_4$.

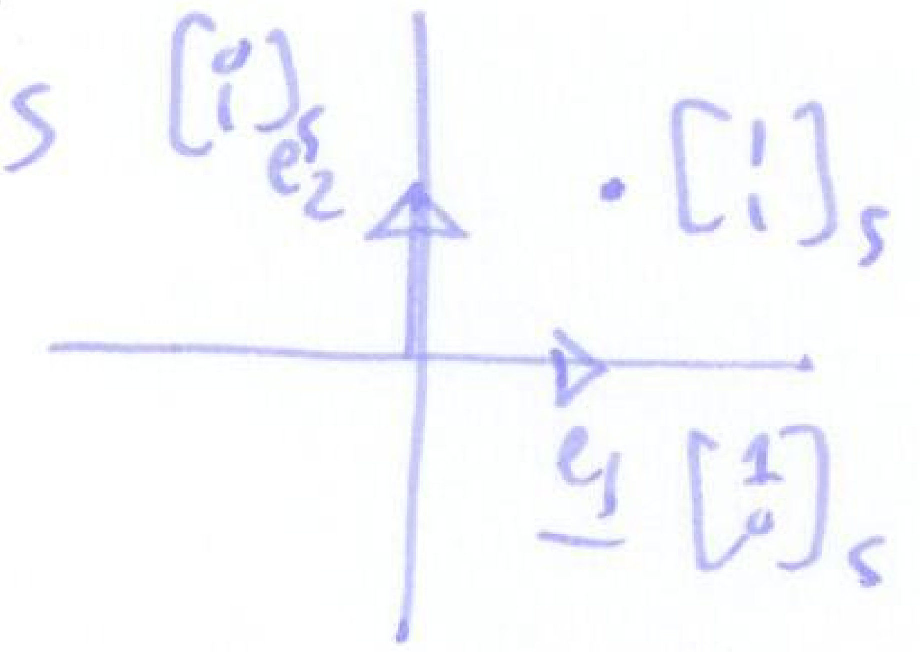
i.e. solve. $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 & -6 \\ 0 & 0 & -1 & 0 & 2 \end{array}\right]$ $\quad$ $\underline{\text{soln}}$ $[\underline{v}]_S = \begin{bmatrix} 3 \\ -1 \\ -2 \\ 1 \end{bmatrix}$.

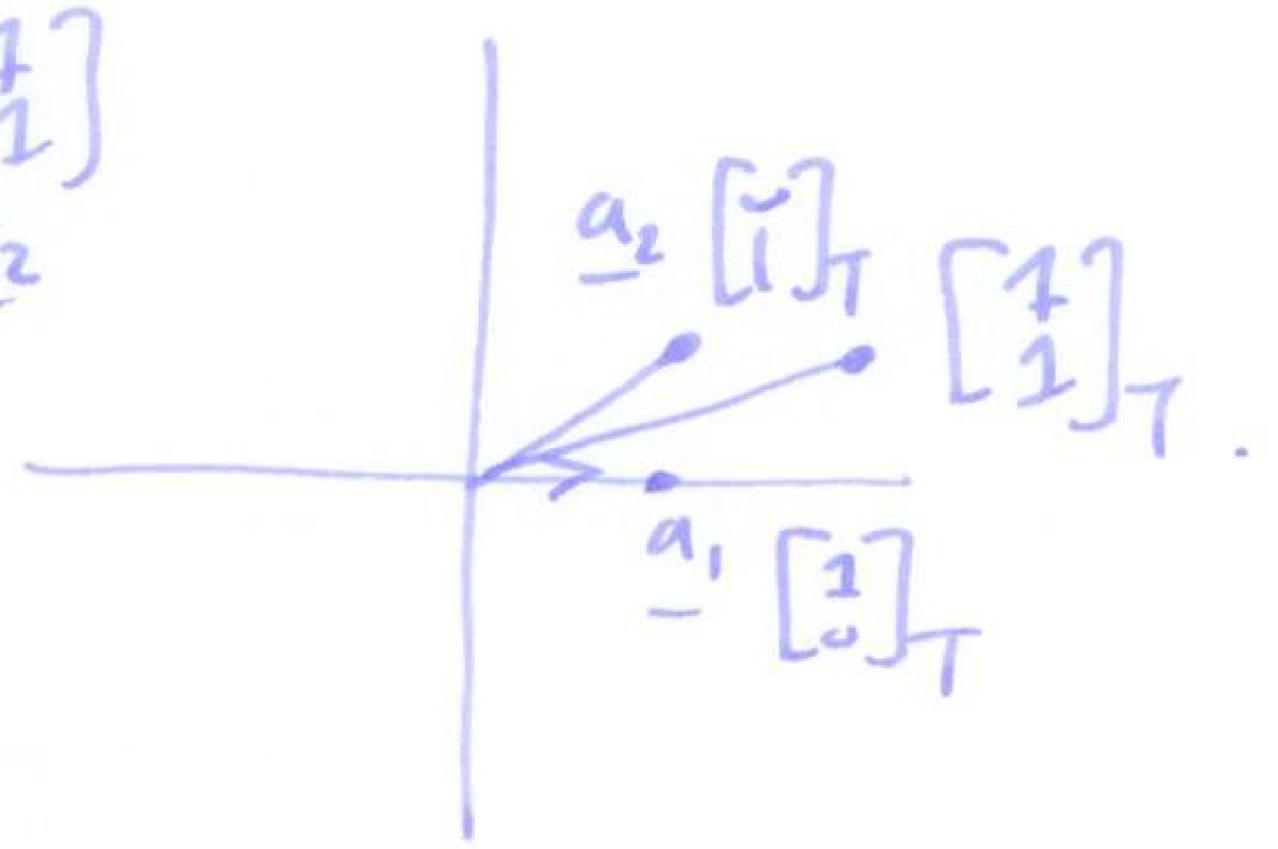
<u>Observation</u> $\quad [\underline{v}]_S + [\underline{w}]_S = [\underline{v}+\underline{w}]_S$

$$[k\underline{v}]_S = k[\underline{v}]_S .$$

<u>Conclusion</u> – a coordinate system on a vector space is a vector space.
– change of coordinates is a linear map.

<u>Pictures</u> $\mathbb{R}^2_S$ $\qquad\qquad\qquad$ $\mathbb{R}^2_T$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \underline{a_1} \ \underline{a_2}$





<u>Transition matrices</u>

$S = \{\underline{v}_1, \dots \underline{v}_n\}$ $\quad T = \{\underline{w}_1, \dots \underline{w}_n\}$ bases.

$\underline{v} \in V$ then $\underline{v} = c_1\underline{w}_1 + c_2\underline{w}_2 + \dots + c_n\underline{w}_n$, so $[\underline{v}]_T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

~~similarly~~, so $\underline{v} \in$.

$$[\underline{v}]_S = [c_1\underline{w}_1 + c_2\underline{w}_2 + \dots + c_n\underline{w}_n]_S$$

$$= c_1[\underline{w}_1]_S + c_2[\underline{w}_2]_S + \dots + c_n[\underline{w}_n]_S$$

so if we know $[\underline{w}_1]_S, \dots [\underline{w}_n]_S$, we can work out $[\underline{v}]_S$. from $[\underline{v}]_T$.

~~to~~ suppose $[\underline{w}_j]_S = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$

Then $[\underline{v}]_S = c_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}$

$$[\underline{v}]_S = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\text{transition matrix.}} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{[\underline{v}]_T.}$$

<u>notation</u>: transition matrix often called $P_{S\leftarrow T}$

$$[\underline{v}]_S = P_{S\leftarrow T} [\underline{v}]_T.$$

the <u>columns</u> of $P_{S\leftarrow T}$ are the vectors $[\underline{w}_i]_S$

i.e. the basis element of T written down in terms of S.

<u>Example</u> $\mathbb{R}^3$: $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ $T = \{\underline{w}_1, \underline{w}_2, \underline{w}_3\}$

$\underline{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\underline{w}_1 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$ $\underline{w}_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$ $\underline{w}_3 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$.

find $P_{S\leftarrow T}$

method ① need to find $\underline{w_i}$ in terms of S.

i.e. solve $\underline{w_1} = c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_3}$, $\underline{w_2} = d_1\underline{v_1} + d_2\underline{v_2} + d_3\underline{v_3}$, etc...

i.e. solve $\left[\underline{v_1}\ \underline{v_2}\ \underline{v_3} \,\middle|\, \underline{w_1}\right]$ i.e. solve $\left[\underline{v_1}\ \underline{v_2}\ \underline{v_3} \,\middle|\, \underline{w_2}\right]$ etc..

so can solve all at once : $\left[\underline{v_1}\ \underline{v_2}\ \underline{v_3} \,\middle|\, \underline{w_1}\ \underline{w_2}\ \underline{w_3}\right]$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 6 & 4 & 5 \\ 0 & 2 & 1 & 3 & -1 & 5 \\ 1 & 0 & 1 & 3 & 3 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Thm $P_{S\leftarrow T}$ is non-singular.

Proof compute $\ker(P_{S\leftarrow T})$ i.e. suppose

$\underline{0} = c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_n}$ but $\{v_1, v_2, v_n\}$ linear independent $\Rightarrow c_1 = c_2 = \cdots = c_n = 0$

so $\ker(P_{S\leftarrow T}) = \{0\}$, then $\operatorname{rank}(P_{S\leftarrow T}) + \underbrace{\operatorname{nullity}(P_{S\leftarrow T})}_{0} = n$.

$\Rightarrow \operatorname{rank}(P_{S\leftarrow T}) = n$.

method ② $E = \{e_1, e_2, e_3\}$ standard basis.

note solve $\underline{v_1} = c_1e_1 + c_2e_2 + c_3e_3$ easy!

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ so } \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underline{v_1}.$$

so $P_{E\leftarrow S} = \left[\underline{v_1}\ \underline{v_2}\ \underline{v_3}\right]$ similarly $P_{E\leftarrow T} = \left[\underline{w_1}\ \underline{w_2}\ \underline{w_3}\right]$

so $P_{S\leftarrow T} = P_{E\leftarrow S}^{-1} P_{E\leftarrow T}$

§6.8 Orthonormal bases in $\mathbb{R}^n$

We'll now consider $\mathbb{R}^n$ with the inner product (dot product).

Defn A set $S = \{\underline{v}_1, \dots \underline{v}_k\}$ in $\mathbb{R}^n$ is <u>orthogonal</u> if $\underline{v}_i \cdot \underline{v}_j = 0$ for all i, j. $i \neq j$.

Example standard basis is orthogonal. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ also $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$ ....

Thm Let $S = \{\underline{v}_1, \dots \underline{v}_k\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Then S is linearly independent.

Proof suppose $c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_k\underline{v}_k = \underline{0}$

then $\underline{v}_i \cdot (c_1\underline{v}_1 + \cdots + c_k\underline{v}_k) = \underline{v}_i \cdot \underline{0}$

$c_i \|\underline{v}_i\|^2 = 0 \Rightarrow c_i = 0.$

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can do this for all i □

Defn $S = \{\underline{v}_1, \dots \underline{v}_k\}$ is <u>orthonormal</u> if it is orthogonal and $\|\underline{v}_i\| = 1$ for all i.

Defn An orthonormal basis for $\mathbb{R}^n$ is an orthonormal set which is a basis.

Thm Let $S = \{\underline{v}_1, \dots \underline{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$, and $\underline{v}$ any vector in $\mathbb{R}^n$ then $\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n$ where $c_i = \underline{v} \cdot \underline{v}_i$.

Proof $\underline{v} = c_1\underline{v_1} + \dots + c_n\underline{v_n}$ for some $c_1 \dots c_n$

$\underline{v}\cdot\underline{v_i} = c_1\underline{v_1}\cdot\underline{v_i} + \dots + c_i\underline{v_i}\cdot\underline{v_i} + \dots + c_n\underline{v_n}\cdot\underline{v_i}$

$\underline{v}\cdot\underline{v_i} = c_i\|v_i\| = c_i$. $\square$

Corollary if $S = \{\underline{v_1}, \dots, \underline{v_n}\}$ is an orthogonal basis, then

$\underline{v} = c_1 v_1 + \dots + c_n v_n$, where $c_i = \dfrac{\underline{v}\cdot\underline{v_i}}{\|v_i\|^2}$ $\square$

how to find an orthonormal basis, given a basis:

Th^m^ (Gram-Schmidt Process)

Given a basis $\{\underline{v_1}, \dots, \underline{v_n}\}$, we can construct an orthonormal basis $\{\underline{w_1}, \dots, \underline{w_n}\}$. first find an orthogonal basis, then normalize.

Proof start with $\underline{w_1} = \underline{v_1}\cdot\dfrac{1}{\|v_1\|}$, then $\|w_1\| = 1$.

now find $\underline{w_2}$ orthogonal to $\underline{w_1}$ in span$\{\underline{v_1}, \underline{v_2}\}$

set $\underline{w_2} = \underline{v_2} + \lambda\underline{v_1}$ want to choose λ s.t. $\underline{w_1}, \underline{w_2}$ orthogonal.

$\underline{w_1}\cdot\underline{w_2} = \underline{v_1}\cdot\underline{v_2} + \lambda\underline{v_1}\cdot\underline{v_1} = 0$

so $\lambda = -\dfrac{\underline{v_1}\cdot\underline{v_2}}{\underline{v_1}\cdot\underline{v_1}}$

i.e. $\underline{w_2} = \underline{v_2} - \dfrac{\underline{v_1}\cdot\underline{v_2}}{\underline{v_1}\cdot\underline{v_1}}\underline{v_1}$.

now find w_3 orthogonal to w_1, w_2 in $\text{span}\{v_1, v_2, v_3\}$

$= \text{span}\{w_1, w_2, v_3\}$.

set $w_3 = v_3 + \lambda_1 w_1 + \lambda_2 w_2$

want $w_1 \cdot w_3 = \cancel{} = w_1 \cdot v_3 + \lambda_1 w_1 \cdot w_1 + \lambda_2 \underbrace{w_2 \cdot w_1}_{0} = 0$

$w_2 \cdot w_3 = w_2 \cdot v_3 + \lambda_1 \underbrace{w_1 \cdot w_2}_{0} + \lambda_2 w_2 \cdot w_2 = 0$.

so $\lambda_1 = -\dfrac{w_1 \cdot v_3}{w_1 \cdot w_1} = -\dfrac{v_1 \cdot v_3}{v_1 \cdot v_1}$.

$\lambda_2 = -\dfrac{w_2 \cdot v_3}{w_2 \cdot w_2}$

so $w_3 = v_3 - \dfrac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \dfrac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$ and so on □.

Finally normalize, replace w_i with $w_i \cdot \dfrac{1}{\|w_i\|}$.