then the vectors $\{\underline{x}_1, \underline{x}_2, \dots \underline{x}_n\}$ are a basis for the solution space.

Defn The dimension of the null space or kernel of A is called the nullity of A.

Non-homogeneous systems

$A\underline{x} = \underline{b}$ corresponding homogeneous system is $A\underline{x} = \underline{0}$.

spose you know a single solution $\underline{x}_p$ to $A\underline{x} = \underline{b}$, i.e. $A\underline{x}_p = \underline{b}$

then any homogeneous solution, $\underline{x}_h$ $(A\underline{x}_h = 0)$ can be added to $\underline{x}_p$ to give a solution to the (non-homogeneous) system.

$$A(x_p + x_h) = Ax_p + Ax_h = \underline{b} + \underline{0} = \underline{b}.$$

i.e. solutions to $A\underline{x} = \underline{b}$ are a parallel copy of solution to $A\underline{x} = \underline{0}$ (but not a vector space)

§6.6 Matrix rank

Defn A $m \times n$ matrix

the rows of A are $(1 \times n)$ vectors – span a subset of $\mathbb{R}^n$ called the row space of A. The columns of A are $(m \times 1)$ vectors, span a subspace of $\mathbb{R}^m$ called the column space.

Thm If A and B are row equivalent, then the row spaces of A and B are equal.

Proof row operations correspond to linear combinations of rows.

$A \xrightarrow[\text{operations}]{\text{row}} B \Rightarrow$ rows of B are linear combinations of rows of A

$\Rightarrow \operatorname{span}(\text{rows } B) \subset \operatorname{span}(\text{rows } A)$

but also $B \xrightarrow[\text{operations}]{\text{row}} A \Rightarrow$ rows of A linear combinations of rows of B

$\operatorname{span}(\text{rows } A) \subset \operatorname{span}(\text{rows } B)$

$\Rightarrow \operatorname{span}(\text{rows } A) = \operatorname{span}(\text{rows } B)$.

Warning row operations change the column space! example $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \overset{\text{row}}{\rightsquigarrow} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

This gives a procedure for finding a basis for the span of a set of vectors $\underline{v}_1, \dots \underline{v}_k$ write $A = \begin{bmatrix} - \underline{v}_1 - \\ \underline{v}_2 \\ \vdots \\ \underline{v}_k \end{bmatrix}$ and row reduce. resulting non-zero vectors are a basis.

Example $A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Note: Leading 1's imply rows are <u>linearly independent</u>.

Defn The dimension of the row (col) space of A is called the <u>row rank</u> (<u>col rank</u>).

Thm Row rank = column rank.

<u>Proof</u> $\underset{\sim}{A} = \begin{bmatrix} | & | & & | \\ \underline{w_1} & \underline{w_2} & \cdots & \underline{w_n} \\ | & | & & | \end{bmatrix}$ find dimension of column space.

consider $c_1\underline{w_1} + c_2\underline{w_2} + \cdots + c_n\underline{w_n} = \underline{0}$ (*)

<u>note</u> : there is a basis for span$\{\underline{w_1}, \ldots, \underline{w_n}\}$ which is a <u>subset</u> of $\{\underline{w_1}, \ldots, \underline{w_n}\}$. How to find this : solve (*) for $c_1 \ldots c_n$.

row reduce $[A \mid \underline{0}]$ get $\left[\begin{array}{ccc|c} 1 \cdots & 0 \cdots & = & \\ 0 & \vdots & = & \\ \vdots & 1 & = & \underline{0} \\ 0 & 0\ 0\ 0 & 0\ 0 & \\ 0 & 0\ 0\ 0\ 0 & 0\ 0 & \end{array}\right]$.

<u>claim</u> vectors corresponding to columns with leading 1's form a basis for span$\{\underline{w_1}, \ldots \underline{w_n}\}$.

<u>recall</u> can assign non-leading one column c_i's arbitrary values, so assign them all zero.

suppose $\underline{w} \in$ span $\underline{w_i}$ then $\underline{w} = c_1\underline{w_1} + c_2\underline{w_2} + \cdots + c_n\underline{w_n}$

assign all non-leading ~~1~~ terms zero, then $\underline{w} = c_1\underline{w_1} + \cdots + c_n\underline{w_n}$ only leading one terms, so $\underline{w_i}$-leading ones are a basis □.

$\therefore$ column rank = # leading ones.
but row rank = # leading ones.

$\Big\}$ equal □.

<u>Defn</u> The <u>rank</u> of a matrix is row rank = column rank.

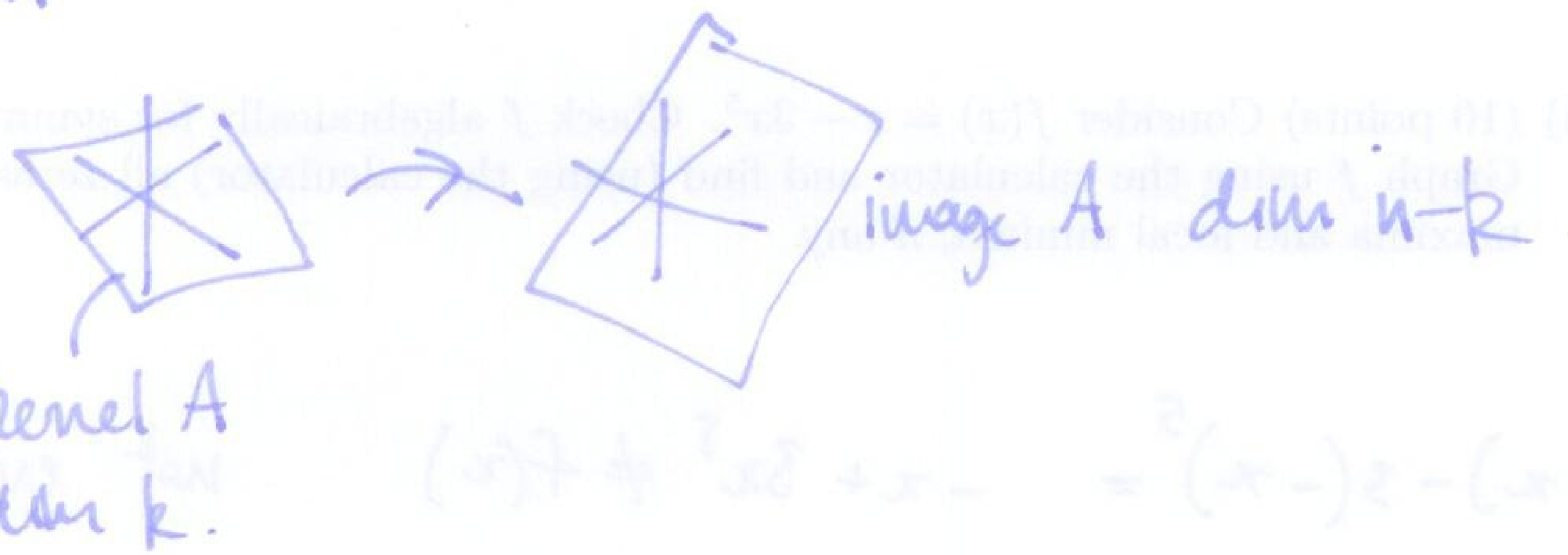
<u>Thm</u> If A is an $(m \times n)$ matrix, then

$$\boxed{\text{rank}\, A + \text{nullity}\, A = n}$$

<u>Proof</u> row reduce : rank = # cols with leading one's.
nullity = # cols <u>without</u> leading one's □.

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

$m \times n$



Rank and singularity

Thm An $(n \times n)$ matrix is nonsingular iff rank $A = n$.

Proof $\Rightarrow$ non-singular $\Rightarrow$ row equivalent to I_n has rank n.

$\Leftarrow$ suppose rank $A = n$ $\Rightarrow$ reduced row echelon form has n leading one's $\Rightarrow$ row reduced form is I_n $\Rightarrow$ non-singular. $\square$.

Corollary A $(n \times n)$ matrix then rank $A = n$ iff $\det(A) \neq 0$.

Corollary A $n \times n$ matrix, then $A\underline{x} = \underline{b}$ has unique solution for each $\underline{b}$ iff rank $A = n$.

Corollary $S = \{\underline{v}_1, \dots, \underline{v}_n\}$ set of n vector in $\mathbb{R}^n$, let $A = [\underline{v}_1 \dots \underline{v}_n]$ then S linearly independent iff $\det(A) \neq 0$.

Corollary $A\underline{x} = 0$ if n equations in n unknowns, then non-trivial solution iff rank $A < n$.