Corollary If $S = \{\underline{v}_1, \dots \underline{v}_n\}$, $T = \{\underline{w}_1, \dots, \underline{w}_m\}$ are basis for V, then $n = m$.

Proof S basis, T linearly independent $\Rightarrow$ $m \leq n$
T basis, S linearly independent $\Rightarrow$ $n \leq m$ $\Big\}$ $\Rightarrow m = n$ $\square$.

Defn The dimension of a vector space V is the number of vectors in a basis for V. Say $\{\underline{0}\}$ has dimension zero.

Examples $\dim \mathbb{R}^n = n$ $\quad\quad \dim C^\infty(\mathbb{R}, \mathbb{R}) = \infty$.
$\dim P_n = n+1$
$\dim M_{mn} = mn$

Thm If S, finite, spans V, then there is a subset of S which is a basis.

Proof Let $S = \{\underline{v}_1, \dots, \underline{v}_k\}$ spans V, suppose linearly dependent, then
$c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k = \underline{0}$ not all $c_i = 0$, wlog (reorder) $c_k \neq 0$

so $\underline{v}_k = \frac{c_1}{c_k}\underline{v}_1 + \dots + \frac{c_{k-1}}{c_k}\underline{v}_k$ so $\underline{v}_1, \dots \underline{v}_{k-1}$ also span V.

continue throwing out vectors till you get a linearly independent set which spans V $\square$. $\boxed{\text{do constructive proof as well}}$

~~Thm If S is a linearly independent set of vectors in a finite dim vector space, then there is a basis which contains S.~~

Thm If $\dim V = n$, and a set of n vectors span V, then they are independent.

~~Proof Let $\{\underline{v}_1, \dots, \underline{v}_n\}$ span V, suppose they are dependent, i.e.~~
~~$c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_n\underline{v}_n = \underline{0}$ not all $c_i = 0$. wlog $c_n \neq 0$ (reorder)~~

~~then $\underline{v}_n = \frac{c_1}{c_n}\underline{v}_1 + \dots + \frac{c_{n-1}}{c_n}\underline{v}_{n-1}$ so any $\underline{w} \in V$ can be written as~~

~~$\underline{w} = d_1\underline{v}_1 + \dots + d_n\underline{v}_n$, so $\underline{w} = d_1\underline{v}_1 + \dots + d_{n-1}\underline{v}_{n-1} + d_n\left(\frac{c_1}{c_n}\underline{v}_1 + \dots + \frac{c_{n-1}}{c_n}\underline{v}_{n-1}\right)$~~

$S = \{\underline{v}_1, \dots \underline{v}_n\}$ there is a finite subset which is a basis for span S.

spose $\underline{v} \in$ span S so $\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_n\underline{v}_n$

how would we find $c_1 \dots c_n$?

solve: $\left[\underline{v}_1\ \underline{v}_2 \dots \underline{v}_n \,\middle|\, \underline{v}\right]$ row reduce:

rearrange cols:

$$\left[\begin{array}{ccccccc|c}
\underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_r & \underline{v}_{r+1} & \dots & \underline{v}_n & \\
1 & 0 & & 0 & \vdots & & \vdots & \vdots \\
0 & 1 & & 0 & \vdots & & \vdots & \vdots \\
 & & \ddots & & & & & \vdots \\
 & & & 1 & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & & & & & & & \\
0 & \dots & & & & & 0 & 0
\end{array}\right]$$

can chose $c_{r+1} \dots c_n$ to be anything (free variable)

so chose them to be zero!

so every $\underline{v} \in$ span S is $\underline{v} = c_1\underline{v}_1 + \dots + c_r\underline{v}_r$

v_i correspond to cols with leading ones.

Proof Suppose $\{v_1, \dots, v_n\}$ span V, then there is a subset which is a basis. wlog $\{\underline{v}_1, \dots \underline{v}_k\}$ but $\dim V = n = k \Rightarrow \{\underline{v}_1, \dots \underline{v}_k\}$ is a basis, hence indep. □.

Thm If S is a linearly independent set of vectors in V, then there is a basis which contains S. (i.e. you can extend a linearly independent set to a basis).

Proof Let $S = \{\underline{v}_1, \dots \underline{v}_k\}$, Let $T = \{\underline{e}_1, \dots \underline{e}_n\}$ be a basis.

consider $\{\underline{v}_1, \underline{e}_1, \dots, \underline{e}_n\}$ linearly dependent as $\underline{v}_1 = c_1\underline{e}_1 + \dots + c_n\underline{e}_n$ not all $c_i = 0$, discard some e_i with $c_i \neq 0$ (wlog $\underline{e}_1$) to get $\{\underline{v}_1, \underline{e}_2, \dots \underline{e}_n\}$ which spans, n elements $\Rightarrow$ linearly independent, so in fact is a basis. Now add $\underline{v}_2$, and so on.... □.

Thm Let V have dimension n, and let $S = \{\underline{v}_1, \dots \underline{v}_n\}$ be a set of n vectors in V. Then

a) if S is linearly independent, then it is a basis.

b) if S spans V, then it is a basis.

Proof (exercise).

§6.5 Homogeneous systems

$A\underline{x} = \underline{0}$ $\quad A$ $(m \times n)$ matrix, i.e. $A: \mathbb{R}^n \to \mathbb{R}^m$.

How to find a basis for the solution space:

solve equations using row reduction.

write answer in form $\underline{x} = s_1\underline{x}_1 + s_2\underline{x}_2 + \dots + s_n\underline{x}_n$