6.3. Linear independence

Q: how do we describe a vector (sub)space?

Defn A set of vectors $S = \{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_k\}$ in a vector space V **span** V if every vector in V is a linear combination of $\underline{v}_1, \ldots, \underline{v}_k$.

write S spans V or $\operatorname{span} S = V$.

Example $\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Q: do $\underline{v}_1, \underline{v}_2, \underline{v}_3$ span $\mathbb{R}^3$?

$\underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ want: $c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 2 & 0 & 1 & y \\ 1 & 2 & 1 & z \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -2 & -1 & y-2x \\ 0 & 1 & 0 & z-x \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 0 & \cancel{y-2x}\ z-x \\ 0 & -2 & 1 & y-2x \end{array}\right]. \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 0 & z-x \\ 0 & 0 & 1 & 2z+y-4x \end{array}\right]$$

YES.

Example show $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ span symmetric matrices $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

Linear independence

Defn a set of vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_k$ is linearly dependent if there exist numbers $c_1, c_2, \ldots c_k$ not all zero such that

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_k\underline{v}_k = \underline{0}$$

A set of vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ is <u>linearly independent</u> if

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_k\underline{v}_k = \underline{0} \Rightarrow c_i = 0 \text{ for all } i.$$

<u>Example</u> are $\begin{bmatrix}-1\\+1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\\1\end{bmatrix}$ linearly independent?

$$c_1\underline{v}_1 + c_2\underline{v}_2 = \underline{0} \qquad c_1\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}-2\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

$$\begin{bmatrix}-1 & -2\\1 & 0\\0 & 1\\0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 0\\-1 & -2\\0 & 1\\0 & 0\end{bmatrix} \quad \begin{bmatrix}1 & 0\\0 & -2\\0 & 1\\0 & 0\end{bmatrix} \quad \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{bmatrix}.$$

<u>Observation</u> every set of vectors containing the zero vector $\underline{0}$ is YES. linearly dependent. i.e. $c_1\underline{0} + \underbrace{(\cdots)}_{\underline{0}} = \underline{0}$. choose $c_1 \neq 0$.

(and not distinct ⇒ dependent)

<u>In $\mathbb{R}^2$</u> : 1 (non-zero) vector always linearly independent

2 (non-zero) vectors dependent if they are parallel
otherwise independent and span $\mathbb{R}^2$.

≥ 3 vectors always dependent.

<u>in $\mathbb{R}^3$</u> : 1 vector always independent.

2 vectors independent iff span a plane.

3 iff span $\mathbb{R}^3$.

≥ 4 dependent.

Thm $\underline{v}_1, \underline{v}_2, \dots \underline{v}_k$ are linearly dependent iff one of the vectors $\underline{v}_i$ $i \geq 2$ is a linear combination of the preceding vectors.

Proof $\Rightarrow$ dependent $\Rightarrow$ $c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k = \underline{0}$

choose $\underline{v}_i$ largest vector with $c_i \neq 0$ then $c_i\underline{v}_i = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_{i-1}\underline{v}_{i-1}$

$$\underline{v}_i = \frac{c_1}{c_i}\underline{v}_1 + \frac{c_2}{c_i}\underline{v}_2 + \dots + \frac{c_{i-1}}{c_i}\underline{v}_{i-1}.$$

$\Leftarrow$

suppose $\underline{v}_i = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_{i-1}\underline{v}_{i-1}$

then choose $c_i = -1$: $c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_{i-1}\underline{v}_{i-1} + c_i\underline{v}_i = \underline{0}$.

§6.4 Basis and dimension

Defn A set of vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k$ in V are a basis for V

if a) $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k$ span V

b) $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k$ are linearly independent in V.

Example $\mathbb{R}^2$: $\underline{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $\underline{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ are a basis for $\mathbb{R}^2$.

similarly $\underline{e}_1, \underline{e}_2, \dots \underline{e}_n$ is a basis for $\mathbb{R}^n$.

Example $\underline{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$ is a basis for $\mathbb{R}^2$.

span $c_1\underline{v}_1 + c_2\underline{v}_2 = \underline{x} = \begin{bmatrix}x\\y\end{bmatrix}$ $\left[\begin{array}{cc|c}2 & 1 & x\\1 & 1 & y\end{array}\right]$ $\left[\begin{array}{cc|c}1 & 1 & y\\2 & 1 & x\end{array}\right]$ $\left[\begin{array}{cc|c}1 & 1 & y\\1 & 0 & x-y\end{array}\right]$

$\left[\begin{array}{cc|c}1 & 1 & y\\0 & -1 & x-2y\end{array}\right]$ span.

independent span $c_1\underline{v}_1 + c_2\underline{v}_2 = \underline{0}$ $\left[\begin{array}{cc|c}2 & 1 & 0\\1 & 1 & 0\end{array}\right]$ $\left[\begin{array}{cc|c}1 & 1 & 0\\0 & -1 & 0\end{array}\right]$.

Def^{n} A vector space is finite-dimensional if it has a finite basis.

Example $f: \mathbb{R} \to \mathbb{R}$ infinite dimensional.

Th^{m} If $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$ is a basis for V, then every vector in V can be written in a unique way as a linear combination of vectors in S.

Proof S basis, so spans, so every vector in V is a linear combination of vectors in S. Now suppose

$$\underline{v} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_n \underline{v}_n$$

$$\underline{v} = d_1 \underline{v}_1 + d_2 \underline{v}_2 + \dots + d_n \underline{v}_n$$

$$\underline{v} - \underline{v} = \underline{0} = (c_1 - d_1)\underline{v}_1 + (c_2 - d_2)\underline{v}_2 + \dots + (c_n - d_n)\underline{v}_n$$

as S linearly independent $\Rightarrow c_i - d_i = 0$ for all i $\Rightarrow c_i = d_i$ for all i. □.

Th^{m} $S = \{\underline{v}_1, \dots \underline{v}_n\}$ non-zero vectors in V which span V. Then some subset of S is a basis for V.

Proof suppose S linearly independent. then S is a basis.

suppose S linearly dependent. Then $\exists$ $c_1 \dots c_n$ not all zero such that

$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_k \underline{v}_k = \underline{0}$ so some $\underline{v}_i$ is a linear combination of the preceding vectors, delete $\underline{v}_i$ from S to get smaller set S'.

claim span $S' = V$

spose $\underline{v} = c_1 \underline{v}_1 + \dots + c_{i-1} \underline{v}_i + c_i \underline{v}_i + c_{i+1} \underline{v}_{i+1} + \dots + c_n \underline{v}_n$.

but $\underline{v}_i = d_1 \underline{v}_1 + \dots + d_{i-1} \underline{v}_{i-1}$

so $\underline{v} = (c_1 + c_i d_1)\underline{v}_1 + \dots + (c_{i-1} + c_i d_{i-1})\underline{v}_i + c_{i+1} \underline{v}_{i+1} + \dots + c_n \underline{v}_n$.

continue until new S' is independent. (stops when S single vector) □.

<u>Thm</u> If $S = \{\underline{v}_1, \dots \underline{v}_n\}$ is a basis for a vector space V, and $T = \{\underline{w}_1, \underline{w}_2, \dots \underline{w}_r\}$ is linearly independent, then $r \leq n$.

<u>Proof</u> consider $T_1 = \{\underline{w}_1, \underline{v}_1, \dots, \underline{v}_n\}$ spans V as S spans V.

in fact $\underline{w}_1 = c_1\underline{v}_1 + \dots + c_n\underline{v}_n$ for some $c_1, \dots c_n$ not all zero as $\underline{w}_1 \neq \underline{0}$, so T_1 is linearly dependent. Pick $\underline{v}_i$ with $c_i \neq 0$

then $\underline{v}_i = \frac{1}{c_i}\underline{w}_1 + \frac{c_1}{c_i}\underline{v}_1 + \dots + \frac{c_n}{c_i}\underline{v}_n$, so $\{\underline{w}_1, \underline{v}_1, \dots \underline{v}_{i-1}, \underline{v}_{i+1}, \dots \underline{v}_n\}$

$= \{\underline{w}_1, \underline{v}_1, \dots \hat{\underline{v}}_i, \dots \underline{v}_n\}$.

still spans V. (notation means throw out $\underline{v}_i$).

now consider $\{\underline{w}_2, \underbrace{\underline{w}_1, \underline{v}_1, \dots \hat{\underline{v}}_i, \dots \underline{v}_n}_{\text{span } V}\}$.

so $\underline{w}_2 = d_1\underline{w}_1 + c_1\underline{v}_1 + \dots + c_n\underline{v}_n$

if all $c_i = 0$ then T is linearly dependent # so some $c_j \neq 0$ – throw this away, then $\{\underline{w}_2, \underline{w}_1, \underline{v}_1, \dots \hat{\underline{v}}_i, \dots \hat{v}_j \dots \underline{v}_n\}$ spans V.

repeat, adding $\underline{w}$'s and discarding $\underline{v}$'s.

If $r \leq n$, no problem.

If $r > n$ then get $\{\underline{w}_n, \underline{w}_{n-1}, \dots, \underline{w}_1\}$ span V

and then $\{\underline{w}_{n+1}, \underline{w}_n, \dots \underline{w}_1\}$ linearly dependent #.

so $r \leq n$ □.