<u>Examples</u>

- $\mathbb{R}^n$
- functions $f : \mathbb{R} \to \mathbb{R}$.

  addition $(f+g)(x) = f(x) + g(x)$.

  $cf(x) = cf(x)$.

- $(m \times n)$ matrices $(A+B)\underline{x} = A\underline{x} + B\underline{x}$.

  $cA\underline{x} = cA\underline{x}$.

- polynomial functions $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$.

- polynomial function of $\deg \leq d$.

<u>Useful facts</u> V vector space.

<u>Thm</u>

a) $0\underline{u} = \underline{0}$ for all $\underline{u} \in V$

b) $c\underline{0} = \underline{0}$ for all $c \in \mathbb{R}$.

c) if $c\underline{u} = \underline{0}$ then either $c = 0$ or $\underline{u} = \underline{0}$.

d) $(-1)\underline{u} = -\underline{u}$ for all $\underline{u} \in V$.

<u>Proof</u> a)

$$0\underline{u} = (0+0)\underline{u} = 0\underline{u} + 0\underline{u}$$

$$0\underline{u} + (-0\underline{u}) = 0\underline{u} + 0\underline{u} + (-0\underline{u})$$

$$\underline{0} = 0\underline{u}.$$

b) <u>note</u>

$$(\cancel{0}+c)\underline{0} = 0.\underline{0} + c\underline{0} + (-c\underline{0}).$$

$$\| \; c\underline{0} + (-c\underline{0})$$

$$\Rightarrow \underline{0} \; \cancel{c\underline{0}} = 0.\underline{0}$$

b) $c\underline{0} = (0+c)\underline{0} = 0.\underline{0} + c\underline{0}$

add $-c\underline{0}$ to both sides: $\underline{0} = 0.\underline{0}$.

now $c\underline{0} = c\,0.\underline{0} = (c0)\underline{0} = 0.\underline{0} = \underline{0}$. □

c) suppose $c\underline{u} = \underline{0}$ and $c \neq 0$

then $\underbrace{\tfrac{1}{c}(c\underline{u})}_{=\underline{u}} = \left(\tfrac{1}{c}\right)\underline{0} = \underline{0}$ so $\underline{u} = \underline{0}$.

d) $(-1)\underline{u} + \underline{u} = (-1)\underline{u} + (1)\underline{u} = (-1+1)\underline{u} = 0\underline{u} = \underline{0}$.

as $-\underline{u}$ unique $-\underline{u} = (-1)\underline{u}$.

§6.2 Subspaces

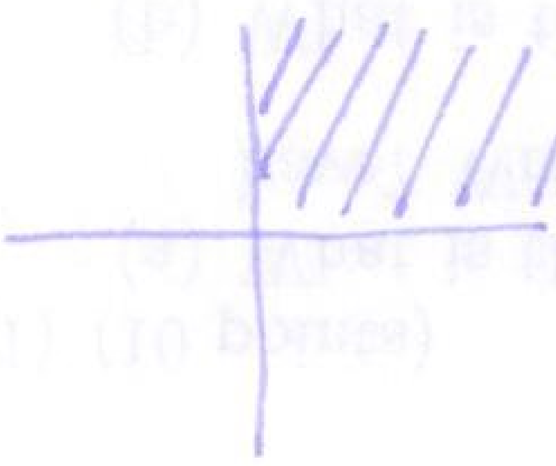
Def^n V vector space. W subset of V.

If W is a vector space with respect to the vector operations in V, then we say W is a subspace of V.

Example $\{\underline{0}\}$ is a subspace of V

V $\qquad$ V.

Nonexample



the quadrant not a vector subspace as $-\begin{bmatrix}1\\1\end{bmatrix}$ not in +ve quadrant.

checking a subset is a subspace:

~~Th~~ **Thm** W subset of V vector space.

Then if α) $\underline{u}, \underline{v}$ vectors in W, then $\underline{u} \oplus \underline{v} \in W$,

and β) $c \in \mathbb{R}$, $\underline{u} \in W$ then $c\underline{u} \in W$,

} i.e. closure properties.

then W is a subspace.

Proof $\square$.

~~Exam~~ Note every subspace contains $\{0\}$.

Example $V = M_{2,3}$ 2x3 ~~to~~ matrices.

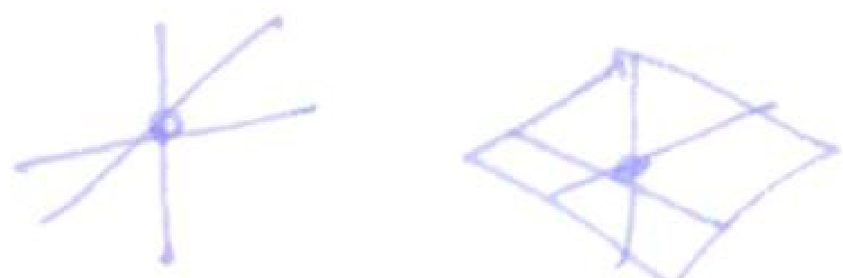
let $W = \begin{bmatrix} 0 & a & b \\ c & d & 0 \end{bmatrix}$ a, b, c, d any real numbers.

check W is a subspace.

α) : $\begin{bmatrix} 0 & a_1 & a_2 \\ a_3 & a_4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b_1 & b_2 \\ b_3 & b_4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 & 0 \end{bmatrix}$.

β) : $c \begin{bmatrix} 0 & a_1 & a_2 \\ a_3 & a_4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ca_1 & ca_2 \\ ca_3 & ca_4 & 0 \end{bmatrix}$.

geometrically



Example $f: \cancel{[a,b]}\ \mathbb{R} \to \mathbb{R}$.

continuous functions $\subset$ all functions.

subspace.

Example consider homogeneous system of equations/linear map $A\underline{x} = \underline{0}$

A mxn $A: \mathbb{R}^n \to \mathbb{R}^m$ let W = solutions of $A\underline{x} = \underline{0}$

claim W is a subspace of $\mathbb{R}^n$

check α) : suppose $\underline{x}, \underline{y}$ solutions $A\underline{x} = \underline{0}$ $A\underline{y} = 0$

then $A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y} = \underline{0} + \underline{0} = \underline{0}$ ✓

β) $A c\underline{x} = c(A\underline{x}) = c\underline{0} = \underline{0}$. ✓.

Defn the solution space of $A\underline{x} = \underline{0}$ is also called the null space.

Note $A\underline{x} = \underline{b}$ ← solutions not a vector subspace if $\underline{b} \neq \underline{0}$!

Constructing subspace

Let $\underline{v}, \underline{w}$ be two vectors in V. Then consider all vectors of the form $a\underline{v} + b\underline{w}$ $a, b \in \mathbb{R}$.

claim this is a subspace of V.

check 1) $(a_1\underline{v} + b_1\underline{w}) + (a_2\underline{v} + b_2\underline{w}) = (a_1 + a_2)\underline{v} + (b_1 + b_2)\underline{w}$.

β) $c(a_1\underline{v} + b_1\underline{w}) = ca_1\underline{v} + cb_1\underline{w}$. ✓.

in general:

Defn Let $\underline{v}_1, \underline{v}_2, \dots \underline{v}_k$ be vectors in a vector space V

a vector $\underline{v} \in V$ is a linear combination of $\underline{v}_1, \underline{v}_2, \dots \underline{v}_k$

if $\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k$ for some numbers $c_1, \dots c_k \in \mathbb{R}$.

Defn Let S be a set of vectors $\{\underline{v}_1, \dots \underline{v}_k\}$

Then span S is the set of all linear combinations of vectors in S.

i.e. $\text{span } S = \{ c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k \mid c_1, \dots c_k \in \mathbb{R} \}$.

Thm Span S is a vector space. $\square$.

Example suppose $\underline{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 5 \\ -5 \\ 2 \end{bmatrix}$ $\underline{v}_4 = \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$

is $\underline{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$?

i.e. can we find numbers c_1, c_2, c_3, c_4 s.t.

$$c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + c_4\underline{v}_4 = \underline{u} .$$

$$c_1\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 5 \\ -5 \\ 2 \end{bmatrix} + c_4\begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 5 & -1 \\ 1 & -2 & -5 & -3 \\ 1 & 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 5 & -1 & 1 \\ 1 & -2 & -5 & -3 & 1 \\ 1 & 0 & 2 & -2 & 2 \end{array}\right] \left[\begin{array}{cccc|c} 1 & -2 & -5 & -3 & 1 \\ 2 & 1 & 5 & -1 & 1 \\ 1 & 0 & 2 & -2 & 2 \end{array}\right] \left[\begin{array}{cccc|c} 1 & -2 & -5 & -3 & 1 \\ 0 & 5 & 15 & 5 & -1 \\ 0 & 2 & 7 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -5 & -3 & 1 \\ 0 & 1 & 3 & 1 & -1/5 \\ 0 & 2 & 7 & 1 & 1 \end{array}\right] \left[\begin{array}{cccc|c} 1 & -2 & -5 & -3 & 1 \\ 0 & 1 & 3 & 1 & -1/5 \\ 0 & 0 & 1 & -1 & 7/5 \end{array}\right]$$ YES.

Example $P_2 = \{a_2x^2 + a_1x + a_0\}$ spanned by $x^2, x, 1$.