Last time n-vectors $\underline{x} \in \mathbb{R}^n$ $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

- addition
- scaler multiplication
- length $\|\underline{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \geq 0$

$$\boxed{\begin{array}{l} \underline{u}\cdot\underline{v} = \underline{v}\cdot\underline{u} \qquad c(\underline{u}+\underline{v}) = c\underline{u} + c\underline{v} \\ (\underline{u}+\underline{v})\cdot\underline{w} = \underline{u}\cdot\underline{w} + \underline{v}\cdot\underline{w} \end{array}}$$

- dot product $\underline{u}\cdot\underline{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$. so $\|\underline{u}\|^2 = \underline{u}\cdot\underline{u}$.

geometric def$^{\underline{n}}$ of dot product.

recall 2/3d: $\underline{u}\cdot\underline{v} = \|\underline{u}\|\|\underline{v}\|\cos\theta$ $\quad \cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\,\|\underline{v}\|}$

claim: this def$^{\underline{n}}$ makes sense in $\mathbb{R}^n$

Wed W2W.
Sun WLO.

Cauchy-Schwartz inequality $|\underline{u}\cdot\underline{v}| \leq \|\underline{u}\|\,\|\underline{v}\|$

Proof true if $\underline{u} = \underline{0}$, so suppose $\underline{u} \neq \underline{0}$.

consider length of $r\underline{u}+\underline{v}$ (r number). $\|r\underline{u}+\underline{v}\| \geq 0$.

$$0 \leq \underbrace{(r\underline{u}+\underline{v})\cdot(r\underline{u}+\underline{v})}_{=\|r\underline{u}+\underline{v}\|^2} = r^2\,\underline{u}\cdot\underline{u} + 2r\,\underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}$$

quadratic in r. non-negative.

$$= r^2\|\underline{u}\|^2 + 2r\,\underline{u}\cdot\underline{v} + \|\underline{v}\|^2.$$

complete the square:

$$\left(r\|\underline{u}\| + \frac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|}\right)^2 + \|\underline{v}\|^2 - \frac{(\underline{u}\cdot\underline{v})^2}{\|\underline{u}\|^2} \geq 0.$$

non-negative $\Rightarrow \|\underline{v}\|^2 - \dfrac{(\underline{u}\cdot\underline{v})^2}{\|\underline{u}\|^2} \geq 0$

$$\Rightarrow (\underline{u}\cdot\underline{v})^2 \geq \|\underline{u}\|^2\|\underline{v}\|^2$$

$$\Rightarrow |\underline{u}\cdot\underline{v}| \geq \|\underline{u}\|\,\|\underline{v}\| \quad \square.$$

now we can define: $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\,\|\underline{v}\|}$

<u>Def$^{\underline{n}}$</u>

$\underline{u}$ and $\underline{v}$ are <u>orthogonal</u> or <u>perpendicular</u> if $\underline{u}\cdot\underline{v} = 0$.

<u>Triangle inequality</u> $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$

<u>Proof</u> $\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})$

$= \underline{u}\cdot\underline{u} + 2\underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}$

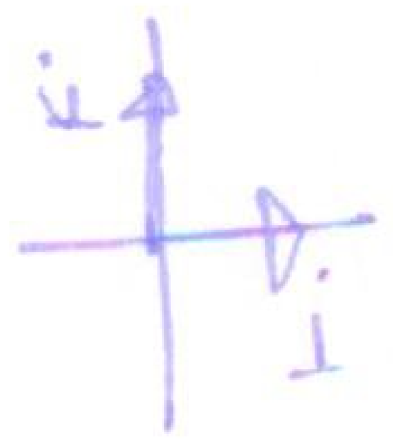
$= \|\underline{u}\|^2 + 2\underline{u}\cdot\underline{v} + \cancel{2u}\,\|\underline{v}\|^2$

Cauchy Schwartz: $\leq \|\underline{u}\|^2 + 2\|\underline{u}\|\|\underline{v}\| + \|\underline{v}\|^2$

$\leq (\|\underline{u}\| + \|\underline{v}\|)^2$ $\square$.

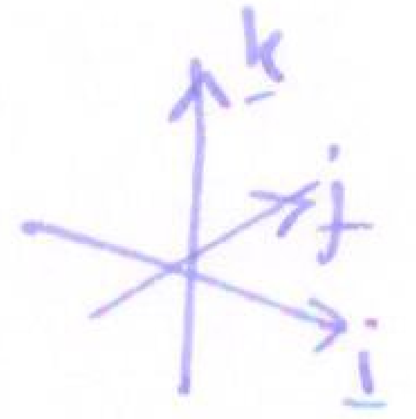
<u>Standard basis vectors</u>

$\mathbb{R}^2$



$\underline{i} = \begin{bmatrix}1\\0\end{bmatrix}$ $\underline{j} = \begin{bmatrix}0\\1\end{bmatrix}$

$\mathbb{R}^3$:



$\underline{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\underline{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\underline{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

in $\mathbb{R}^4$ $\underline{e_1} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $\underline{e_2} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ $\underline{e_3} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ $\underline{e_4} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ $\mathbb{R}^5$ $\underline{e_1} \ldots \underline{e_5}$.

§4.3 Linear transformations

A <u>linear transformation/map/function</u> is a function $L : \mathbb{R}^n \to \mathbb{R}^m$

$$\underline{u} \mapsto L(\underline{u})$$

such that a) $L(\underline{u}+\underline{v}) = L(\underline{u}) + L(\underline{v})$

b) $L(k\underline{u}) = kL(\underline{u})$

for all vectors $\underline{u}, \underline{v} \in \mathbb{R}^n$.

<u>Example</u> $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 4x$ — <u>linear</u>

$(x+y) \mapsto 4(x+y) = 4x+4y = f(x)+f(y)$

$f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ — <u>not</u> linear.

$f(x+y) = (x+y)^2 = x^2+2xy+y^2 \neq f(x)+f(y) = x^2+y^2$.

$x \mapsto x+1$.

<u>Example</u> Let A be an $m \times n$ matrix. Then A defines a function $L : \mathbb{R}^n \to \mathbb{R}^m$, $\underline{u} \mapsto A\underline{u}$ by $L(\underline{u}) = A\underline{u}$

$$\underbrace{(m\times n)(n\times 1)}_{(m\times 1)}.$$

<u>Claim</u> multiplying by a matrix A is a <u>linear map</u>.

check: a) $L(\underline{u}+\underline{v}) = A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v} = L(\underline{u}) + L(\underline{v})$ ✓.

b) $L(c\underline{u}) = Ac\underline{u} = c(A\underline{u}) = cL(\underline{u})$ ✓.

useful facts about linear maps:

Thm if $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map then

$$L(c_1\underline{u}_1 + c_2\underline{u}_2 + \dots + c_k\underline{u}_k) = c_1 L(\underline{u}_1) + c_2 L(\underline{u}_2) + \dots + c_k L(\underline{u}_k).$$

Proof $L(c_1\underline{u}_1 + c_2\underline{u}_2 + \dots + c_k\underline{u}_k) = L(c_1\underline{u}_1) + L(c_2\underline{u}_2) + \dots + L(c_k\underline{u}_k).$

$$= c_1 L(\underline{u}_1) + c_2 L(\underline{u}_2) + \dots + c_k L(\underline{u}_k). \quad \square$$

Thm $L: \mathbb{R}^n \to \mathbb{R}^m$ linear map then a) $L(\underset{\mathbb{R}^n}{\underline{0}}) = \underset{\mathbb{R}^m}{\underline{0}}$!

b) $L(\underline{u} - \underline{v}) = L(\underline{u}) - L(\underline{v})$

Proof a) $L(0 \cdot \underline{u}) = 0 L(\underline{u}) = \underline{0}$.

b) $L(\underline{u} - \underline{v}) = L(\underline{u} + (-1)\underline{v}) = L(\underline{u}) - L(\underline{v}).$ $\square$.

Q: every matrix gives a linear map, does every linear map come from a matrix?

Thm Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then there is a unique $m \times n$ matrix A such that $L(\underline{x}) = A\underline{x}$.

Proof $$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

$$= x_1\underline{e}_1 + x_2\underline{e}_2 + \dots + x_n\underline{e}_n$$

$$L(\underline{x}) = L(x_1\underline{e}_1 + x_2\underline{e}_2 + \dots + x_n\underline{e}_n)$$
$$= x_1 L(\underline{e}_1) + x_2 L(\underline{e}_2) + \dots + x_n L(\underline{e}_n)$$

let A be the $m \times n$ matrix whose i-th column is $L(\underline{e_i})$.

$$A\underline{x} = \begin{bmatrix} | & | & & | \\ L(\underline{e_1}) & L(\underline{e_2}) & \cdots & L(\underline{e_n}) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A(x_1\underline{e_1} + x_2\underline{e_2} + \cdots + x_n\underline{e_n}).$$

$$= x_1 A\underline{e_1} + x_2 A\underline{e_2} + \cdots + x_n A\underline{e_n}$$

$$= x_1 \operatorname{col}_1(A) + x_2 \operatorname{col}_2(A) + \cdots + x_n \operatorname{col}_n(A).$$

$$= x_1 L(\underline{e_1}) + x_2 L(\underline{e_2}) + \cdots + x_n L(\underline{e_n}).$$

$$= L(\underline{x}).$$ as required $\square$.

check : A is unique.

suppose $L(\underline{x}) = B\underline{x}$ for some matrix B.

then $\left.\begin{array}{l} L(\underline{e_i}) = A\underline{e_i} = \operatorname{col}_i(A). \\ L(\underline{e_i}) = B\underline{e_i} = \operatorname{col}_i(B) \end{array}\right\}$ A, B have same columns so $A = B$. $\square$.

$\underline{e_1}, \ldots, \underline{e_n}$ standard basis

$A = [L(\underline{e_1})\ L(\underline{e_2}) \cdots L(\underline{e_n})]$ standard matrix representing L.

Example $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x+y \\ y-z \\ x+z \end{bmatrix}$$ check linear?

find matrix $L(\underline{e_1}) = L\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $L(\underline{e_2}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $L(\underline{e_3}) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

so $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y-z \\ x+z \end{bmatrix}$.

§6.1 Vector spaces

example : $\mathbb{R}^n$ is a vector space.

Def^n a vector space is a set V together with operations

addition : $\oplus : V \times V \longrightarrow V$
$\qquad u, v \longmapsto u \oplus v$

scalar multiplication $\odot$: $\mathbb{R} \times V \longrightarrow V$
$\qquad c, u \longmapsto c \odot \underline{u}$

with the following properties :

a) $\underline{u} \oplus \underline{v} = \underline{v} \oplus \underline{u}$ for all $\underline{u}, \underline{v} \in V$

b) $\underline{u} \oplus (\underline{v} \oplus \underline{w}) = (\underline{u} \oplus \underline{v}) \oplus \underline{w}$ for all $\underline{u}, \underline{v}, \underline{w} \in V$

c) there is an element $\underline{0} \in V$ such that $\underline{0} \oplus \underline{u} = \underline{u} \oplus \underline{0} = \underline{u}$.

d) for each $\underline{u} \in V$ there is an element $-\underline{u}$ such that $\underline{u} + (-\underline{u}) = \underline{0}$.

e) $c \odot (\underline{u} \oplus \underline{v}) = c \odot \underline{u} \oplus c \odot \underline{v}$

f) $(c+d) \odot \underline{u} = c \odot \underline{u} \oplus d \odot \underline{u}$.

g) $c \odot (d \odot u) = (cd) \odot \underline{u}$

h) $1 \odot \underline{u} = \underline{u}$.

elements of V are vectors

$\oplus$ vector addition

$\odot$ scaler multiplication

$\underline{0}$ is called zero vector (unique)

$-\underline{u}$ negative of $\underline{u}$. (unique)

closure $u \oplus v$ is always a vector $c \odot \underline{u}$ is always a vector.