so $\det(A) = a_{11}a_{22}\cdots a_{nn}$. $\square$.

<u>Example</u>

$$\det\begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix} = 6.$$

<u>Corollary</u> determinant of a diagonal matrix is the product of the diagonal elements.

<u>notation for row operations</u>

	$r_i \leftrightarrow r_j$	$c_i \leftrightarrow c_j$
swap	rows	cols.
	$kr_i \to r_i$	$kc_i \to c_i$
multiply by $k \neq 0$.		
	$kr_i + r_j \to r_j$	$kc_i + c_j \to c_j$
add a multiple.		

<u>Practical method to compute $\det(A)$</u>

do row operations on A to make it upper triangular.

<u>Example</u>

$$\begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad \tfrac{1}{2}r_3 \to r_3 \quad 2\begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

swap 1,3 $\quad r_1 \leftrightarrow r_3$.

$$-2\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{bmatrix} \qquad \begin{matrix} r_2 - 3r_1 \to r_2 \\ r_3 - 4r_1 \to r_3 \end{matrix} \quad -2\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{bmatrix}$$

$-4r_2 \to r_2$
$-5r_3 \to r_3$ $\quad -40\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ swap 2, 3 $r_2 \leftrightarrow r_3$ $40\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$r_3 - 2r_2 \to r_3$ $\quad 40\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} = -120.$

Thm $\det(AB) = \det(A)\det(B)$.

Proof maybe later... $\square$.

Corollary if A is nonsingular then $\det(A) \neq 0$.

Proof A non-singular $\Rightarrow \exists A^{-1}$ s.t. $AA^{-1} = I$

$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$.

$\Rightarrow \det(A) \neq 0$

$\Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$. $\square$.

§3.2 Cofactor expansion and applications

Alternate method for computing determinants: (cofactor expansions).

Defn A $n\times n$ matrix

Let M_{ij} be the $(n-1)\times(n-1)$ matrix obtained by deleting the i-th row and j-th column of A. This is called the minor of a_{ij}. The cofactor A_{ij} of a_{ij} is $A_{ij} = (-1)^{i+j}\det(M_{ij})$

Example $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$

$M_{11} = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$ $A_{11} = 12$

$M_{22} = \begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix}$ $A_{22} = -40$.

signs $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Thm A $n \times n$ matrix

row expansion: $\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$

col expansion: $\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$. §

Proof no proof $\square$.

Example find determinant of $\begin{bmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{bmatrix}$.

expand along 2nd col or 3rd row.

Example upper triangular: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ $\det = 1.2.3.4$.

Application to inverses

recall $a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \det(A)$

consider: $a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = 0$ if $i \neq k$.

claim

Proof construct a matrix B from A by replacing k-th row of A by i-th row. Then B has two rows the same, so $\det(B) = 0$.

Now expand B out along k-th row, get: $a_{11}A_{k1} + a_{12}A_{k2} + \cdots + a_{in}A_{kn} = \det(B) = 0$. $\square$.

Def^{n} A $n \times n$ matrix. The adjoint matrix $\operatorname{adj} A$ is the matrix

s.t. $[\operatorname{adj} A_{ij}] = $ cofactor A_{ji} of a_{ji}.

i.e. the adjoint is the transpose of the matrix of cofactors.

Th^{m} $A(\operatorname{adj} A) = \det(A) I_n = (\operatorname{adj} A) A$.

Proof

$$A(\operatorname{adj} A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ \vdots & & & \\ A_{1n} & A_{2n} & & A_{nn} \end{bmatrix}.$$

$$\text{row}_i \cdot \text{col}_j = a_{i1}A_{j1} + a_{i2} + A_{j2} + \cdots + a_{n2}A_{n2} = \begin{cases} 0 & i \neq j \\ \det(A) & i = j \end{cases}.$$

$$= \begin{bmatrix} \det(A) & & 0 \\ & \det(A) & \\ 0 & & \det(A) \end{bmatrix} = \det(A)\, I$$

Corollary $A^{-1} = \dfrac{1}{\det(A)} (\operatorname{adj} A)$

Th^{m} A is non-singular $\Leftrightarrow \det(A) \neq 0$. $\square$

Corollary A $n \times n$ matrix. $A\underline{x} = \underline{0}$ has a non-trivial sol^{n} iff $\det(A) = 0$. $\square$

The following are equivalent:

1. A is non-singular
2. $\underline{x} = \underline{0}$ is the only solution to $A\underline{x} = \underline{0}$
3. A is row equivalent to I_n
4. $A\underline{x} = \underline{b}$ has a unique solution for each $\underline{b}$
5. $\det(A) \neq 0$.

Cramer's rule

linear system $A\underline{x} = \underline{b}$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} & b_n \end{array}\right].$$

if $\det(A) \neq 0$ then

$$\underline{x} = A^{-1}\underline{b} = \begin{bmatrix} \frac{A_{11}}{\det(A)} & \frac{A_{21}}{\det(A)} & \cdots & \frac{A_{n1}}{\det(A)} \\ \vdots & & & \vdots \\ \frac{A_{n1}}{\det(A)} & \cdots & \cdots & \frac{A_{nn}}{\det(A)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

i.e. $$x_i = \frac{A_{1i}}{\det(A)} b_1 + \frac{A_{2i}}{\det(A)} + \cdots + \frac{A_{ni}}{\det(A)} b_n$$

let $$A_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i-1} & b_1 & a_{1i+1} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & & & \\ a_{n1} & a_{n2} & & a_{ni-1} & b_n & a_{ni+1} & \cdots & a_{nn} \end{bmatrix}.$$

replace i-th col with $\underline{b}$

then $\det(A_i) = A_{1i}b_1 + A_{2i}b_2 + \cdots + A_{ni}b_n$

(expand along ith col).

so $\boxed{x_i = \dfrac{\det(A_i)}{\det(A)}}$

Point: sol^ns to $Ax = b$ depend cts on A.

§3.3 Computational view

solve $\underline{A}\underline{x} = \underline{b}$ where A is 25×25 matrix.

sol^n: $\underline{x} = A^{-1}\underline{b}$

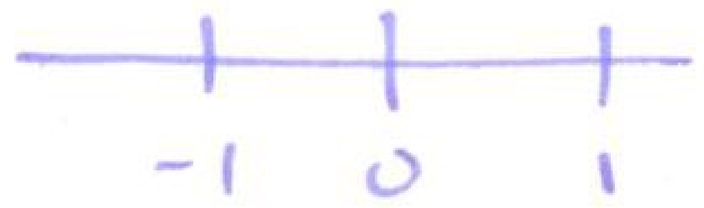
find A^{-1} by cofactors requires $n!$ operations.

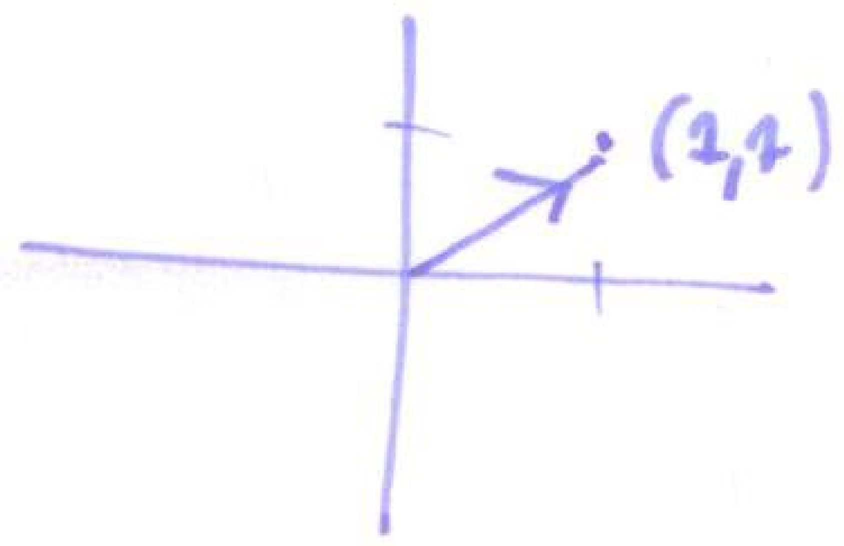
$25! \simeq 10^{25}$

sol^n: $\underline{x} = A^{-1}b$ Gaussian reduction takes $\frac{1}{3}(25)^3$ operations.

(very fast)

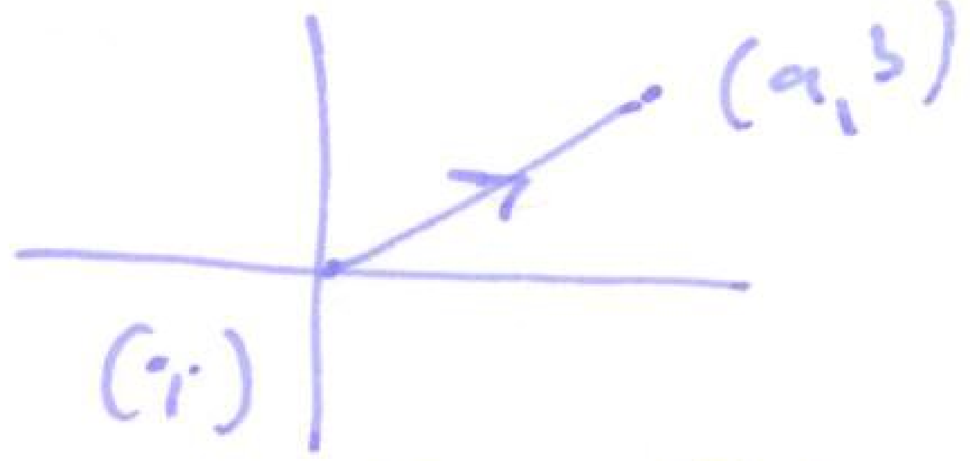
§4.1 Vectors in the plane

$\mathbb{R}$:


$\mathbb{R}^2$


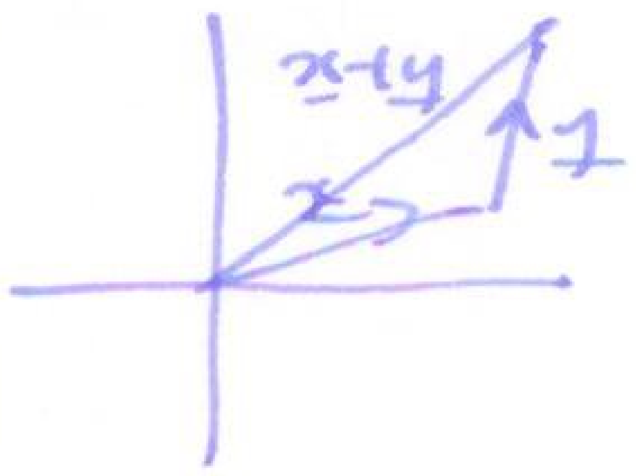
vector : geometrically : directed line segment / length + direction.

coordinates : $\underline{x} = \begin{bmatrix} a \\ b \end{bmatrix}$



length : $\|\underline{x}\| = \sqrt{a^2+b^2}$ (Euclidean length from Pythagoras)

adding vectors : $\underline{x}+\underline{y}$ geometrically :



coordinates $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}$

scalar multiplication $\underline{x}$ vector c number.

$c\underline{x}$ is vector in same direction with length $c\|\underline{x}\|$ if $c>0$
opposite $|c|\|\underline{x}\|$ $c<0$

coordinates : $c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$.

dot product $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\underline{u}\cdot\underline{v} = u_1v_1 + u_2v_2$.

geometric defn $\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta$

angle between two vectors is $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\,\|\underline{v}\|}$.

<u>orthogonal vectors</u> $\underline{u}\cdot\underline{v} = 0$ (i.e. $\theta = \frac{\pi}{2}$).

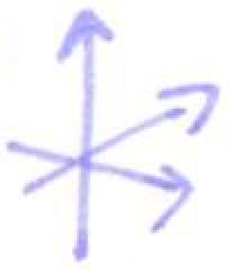
useful properties of dot product:

- $\underline{u}\cdot\underline{u} \geqslant 0$ $\quad \underline{u}\cdot\underline{u} = 0$ iff $\underline{u} = \underline{0}$
- $\underline{u}\cdot\underline{v} = \underline{v}\cdot\underline{u}$.
- $(\underline{u}+\underline{v})\cdot\underline{w} = \underline{u}\cdot\underline{w} + \underline{v}\cdot\underline{w}$.
- $c(\underline{u}\cdot\underline{v}) = (c\underline{u})\cdot\underline{v} = \underline{u}\cdot(c\underline{v})$.

<u>Unit vectors</u> $\quad \underline{u}$ is a <u>unit vector</u> if $\|\underline{u}\| = 1$.

$\underline{v}$ any vector $(\neq \underline{0})$ then $\frac{1}{\|\underline{v}\|}\underline{v}$ is a unit vector.

<u>§4.2 n-vectors</u>

$\mathbb{R}^3$  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad \mathbb{R}^4$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $\quad \mathbb{R}^n$ $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$

two vectors are equal iff
- same dimension
- all components equal.

$\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

<u>addition</u>

$\underline{u}+\underline{v} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$

<u>scalar multiplication</u> $\quad c\underline{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$

(same rules as $\mathbb{R}^2$)

<u>dot products</u>: $\underline{u}\cdot\underline{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}\cdot\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$.

aka (standard) inner product