inverse permutations

$1234 \longmapsto 2143$
$\longleftarrow$

there is an inverse permutation which gets you back to where you start

$\overset{1234}{\cancel{2341}} \longmapsto 23\overset{1}{\cancel{3}}4.$

$\overset{1234}{\cancel{3124}} \longmapsto 3124.$

the inverse permutation has the same even/odd parity as the initial one.

Proof reverse the swaps! (doesn't change number of swaps. □.

matrix notation.

$1234 \longmapsto 2143$

$a_{\underline{1}2}\, a_{\underline{2}1}\, a_{\underline{3}4}\, a_{43}.$

$a_{2\underline{1}}\, a_{1\underline{2}}\, a_{4\underline{3}}\, a_{3\underline{4}}$

$1234 \longmapsto 2314$

$a_{\underline{1}2}\, a_{\underline{2}3}\, a_{\underline{3}1}\, a_{\underline{4}4}$

$a_{3\underline{1}}\, a_{1\underline{2}}\, a_{2\underline{3}}\, a_{44}$

Properties of determinants.

$A = [a_{ij}]$ set $B = A^T$ so $B = [b_{ij}]$ where $b_{ij} = a_{ji}$.

Thm $\det(A^T) = \det(A)$.

Proof $\det(A^T) = \sum (\pm)\, b_{1j_1} b_{2j_2} \cdots b_{nj_n}.$

$= \sum (\pm)\, a_{j_1 1} a_{j_2 2} \cdots a_{j_n n}$

$= \sum (\pm)\, a_{1j_1'} a_{2j_2'} \cdots a_{nj_n'} \longleftarrow$ where j_n' is the inverse permutation — sign stays the same!

$= \det(A)$

Thm If B is obtained from A by multiplying a row (col) by ~~zero~~ a number ~~c~~ c then $\det(B) = c\det(A)$.

Proof
$$\det(B) = \sum (\pm)\, b_{1j_1} \cdots b_{rj_r} \cdots b_{nj_n}$$

$$= \sum (\pm)\, a_{1j_1} \cdots c a_{rj_r} \cdots b_{nj_n}.$$

$$= \sum c(\pm)\, a_{1j_1} \cdots a_{rj_r} \cdots b_{nj_n}$$

$$= c\det(A). \ \square.$$

Example
$$\cancel{\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}} \quad \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

note
$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = 2.2.\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Thm If B is obtained from A by adding c times the s-th row to the r-th row, $(r \neq s)$ then $\det(B) = \det(A)$.

Proof $B = [b_{ij}]$ $A = [a_{ij}]$

where $b_{ij} = a_{ij}$ if $i \neq r$

$b_{rj} = a_{rj} + c\,a_{sj}$.

$$\det(B) = \sum (\pm)\, b_{1j_1} \cdots b_{rj_r} \cdots b_{nj_n}$$

$$= \sum (\pm)\, a_{1j_1} \cdots (a_{rj_r} + c a_{sj_r}) \cdots a_{nj_n}.$$

$$= \underbrace{\sum (\pm)\, a_{1j_1} \cdots a_{rj_r} \cdots a_{nj_n}}_{\det(A)} + c \underbrace{\sum a_{1j_1} \cdots a_{sj_r} \cdots a_{sj_s} \cdots a_{nj_n}}$$

det of:

rows r, s the same!

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{s1} & \cdots & & a_{sn} \\ \vdots & & & \\ a_{s1} & \cdots & & a_{sn} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix}.$$

so $\det = 0$.

$\square$.

<u>Example</u> ~~det~~ $\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -2.$

<u>Thm</u> If A is upper triangular (lower) then $\det(A) = a_{11} a_{22} \cdots a_{nn}$.

<u>Proof</u> $\begin{bmatrix} a_{11} & a_{12} & \cdots \\ 0 & a_{22} & \cdots \\ 0 & & a_{nn} \end{bmatrix}$ $\det(A) = \underset{\text{permutations}}{\sum} (\pm)\, a_{1j_1} \cdots a_{nj_n}.$

spose some $j_i < i \Rightarrow a_{ij_i} = 0$.

$\Rightarrow$ non-zero terms in sum have $j_i \geq i$ for all i.

but then $j_n = n$, $j_{n-1} = n-1 \ldots$ etc. $\Rightarrow j_i = i$ for all i

only 1 permutation id. permutation.

so $\det(A) = a_{11} a_{22} \cdots a_{nn}$. $\square$.

<u>Example</u>

$$\det \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix} = 6.$$

<u>Corollary</u> determinant of a diagonal matrix is the product of the diagonal elements.

<u>notation for row operations</u>

$r_i \leftrightarrow r_j$ $\quad$ $c_i \leftrightarrow c_j$

swap rows $\quad$ cols.

$k r_i \xrightarrow{} r_i$ $\quad$ $k c_i \xrightarrow{} c_i$

multiply by $k \neq 0$.

$k r_i + r_j \to r_j$ $\quad$ $k c_i + c_j \to c_j$

add a multiple.

<u>Practical method to compute det(A)</u>

do row operations on A to make it upper triangular.

<u>Example</u>

$$\begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{bmatrix} \quad \tfrac{1}{2} r_3 \to r_3 \quad 2 \begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

swap 1,3 $\quad$ $r_1 \leftrightarrow r_3$.

$$-2 \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{bmatrix} \quad \begin{matrix} r_2 - 3r_1 \to r_2 \\ r_3 - 4r_1 \to r_3 \end{matrix} \quad -2 \begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{bmatrix}$$