Useful fact if A is a non-singular $n \times n$ matrix, then every system of equations $A\underline{x} = \underline{b}$ has a unique solution.

Proof $A^{-1}A\underline{x} = A^{-1}\underline{b}$

$$\underline{x} = A^{-1}\underline{b}. \quad \square$$

Summary (properties of non-singular matrices).

The following are equivalent:

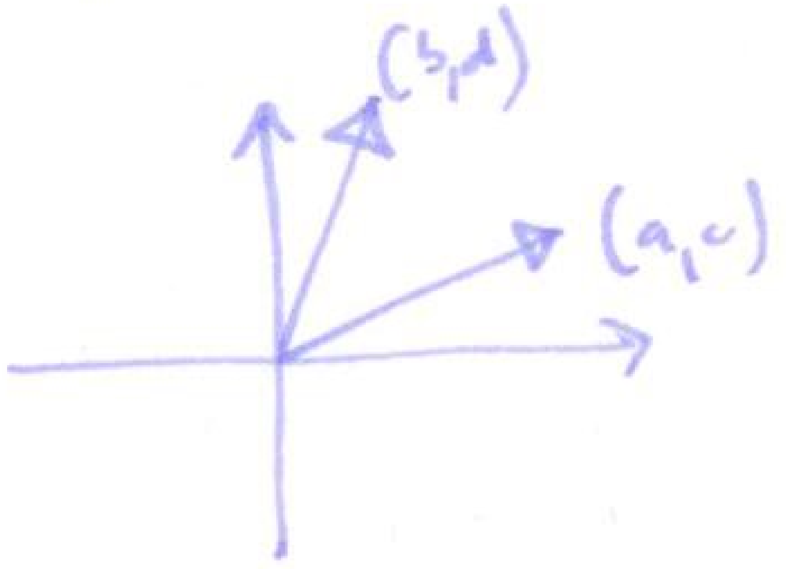
1. A is non-singular.
2. $\underline{x} = \underline{0}$ is the only solution to $A\underline{x} = \underline{0}$
3. A is row equivalent to I_n.
4. $A\underline{x} = \underline{b}$ has a unique solution.

§3.1 Determinants Geometric definitions

Determinant as volume

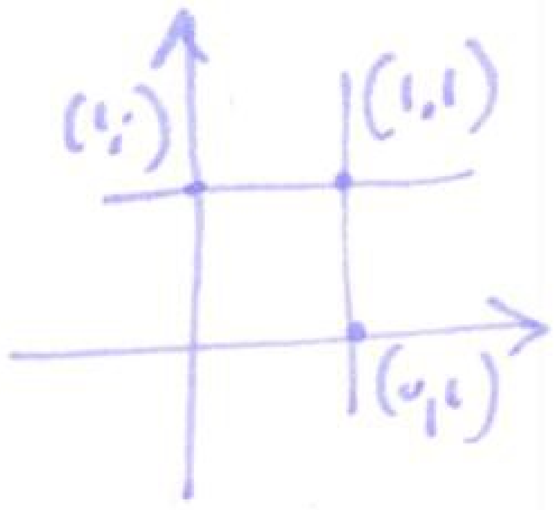
The determinant is a number defined for a square $(n \times n)$ matrix.

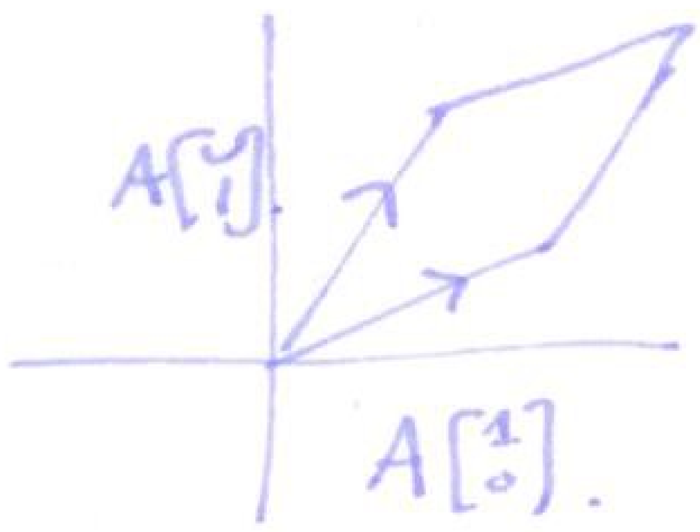
2x2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$



can think of this matrix as two column vectors $\begin{bmatrix} a \\ c \end{bmatrix}$ $\begin{bmatrix} b \\ d \end{bmatrix}$.

why? consider $A : \mathbb{R}^2 \to \mathbb{R}^2$

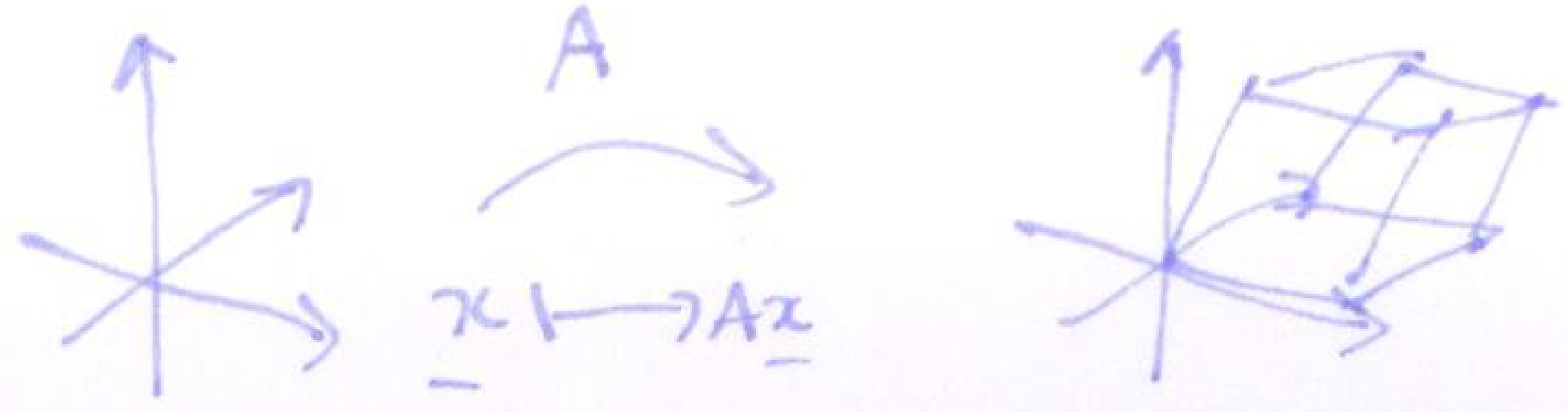




determinant of A, $\det(A)$ = area of parallelogram.

note may be zero! (if column vectors are parallel).

3x3



$\det(A)$ = volume of parallelepiped.

analogous statements in higher dimension.

<u>Algebraic definition</u> (2x2) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(A) = ad - bc$.

(3x3) $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge).$$

<u>Q</u>: how to generalize this to $n \times n$ matrices?

<u>Def</u>: A <u>permutation</u> of the set $\{1, 2, \dots n\}$ consists of the elements of the set in a different order.

<u>Examples</u>. $(1,2)$. $(1,2,3)$. $(2,1,3)$ $(3,1,2)$
$(2,1)$ $(1,3,2)$ $(2,3,1)$ $(3,2,1)$

$(1,4,2,3,5)$ is a permutation of $\{1,2,3,4,5\}$.

<u>Q</u>: how many permutations?

n	1	2	3	4	5	6
# permutations	1	2	6	? 24	120	720.

$\{1, \dots n\}$

$(\underset{\substack{\uparrow \\ n\text{-choices}}}{\sqcup}, \underset{\substack{\uparrow \\ (n-1)\text{ choices}}}{\sqcup}, \dots, \underset{\substack{\uparrow \\ 1\text{-choice.}}}{\sqcup})$

<u>$n!$ permutations</u>.

<u>Even and odd permutations.</u>

A <u>swap</u> is a special permutation that only changes two elements.

example $12345 \longmapsto 14325$.

Useful facts

1. every permutation can be written as a sequence of swaps.
2. if a permutation is a product of an even/odd number of swaps we say it is an even/odd permutation.
~~if a~~
3. every permutation is either even ~~and~~ or odd.
4. there are $\frac{n!}{2}$ even/odd permutions.

Defn $A = [a_{ij}]$ $n \times n$ matrix. The determinant of A, $\det A$, $|A|$.

$$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n}.$$

take sum over all permutations $(j_1, j_2 \ldots j_n)$ of $(1, 2, \ldots n)$

the sign is + if the permutation is even
$\qquad\qquad$ − $\qquad\qquad\qquad\qquad\quad$ odd.

Example $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\det(A) = \sum (\pm) a_{1j_1} a_{2j_2}$

permutions: (12)
(21)

$\det(A) = \cancel{ad}\; a_{11} a_{22} - a_{12} a_{21}$

exercise check 3x3.

note $n!$ gets big – impractical for computing large determinants.