③ add a multiple of one row to another row

$$A \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{bmatrix} \qquad A' : \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{r1}x_1 + \cdots + a_{rn}x_n = b_r \\ + c a_{s1}x_1 + \cdots + c a_{sn}x_n \quad + c b_s . \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m . \end{bmatrix}$$

$x_1, \ldots x_n$ satisfy $A \Rightarrow x_1, \ldots x_n$ satisfy A'!

$$A \xrightarrow{(r) + c(s)} A' \xrightarrow{(r) - c(s)} A'' = A .$$

$x_1, \ldots x_n$ satisfy $A' \Rightarrow x_1, \ldots x_n$ satisfy $A'' = A$. □.

<u>Examples</u> of final step (finding solutions / back substitution).

$$\left[\begin{array}{cccc|c} & \cdots & & & \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

no solutions!.

↑ any non-zero term here.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

unique solution

$x_3 = 1$
$x_2 = 2$
$x_1 = 3$.

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

many solutions.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$x_3 + x_4 - x_5 = 1$$

two "free variables".

chose $x_4 = r$
$x_5 = s$.

$$x_3 = 1 - r + s.$$

$$x_2 + x_4 + x_5 = 1$$

$$x_2 + r + s = 1. \qquad x_2 = 1 - r - s.$$

$x_4 = 1$ $\qquad$ $x_4 = 1$

$x_2 = 1 - r - s$

$x_3 = 1 - r + s.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + r \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

(can easily see 2-parameter 2d solution set).

Homogeneous systems

special case $A\underline{x} = \underline{0}$

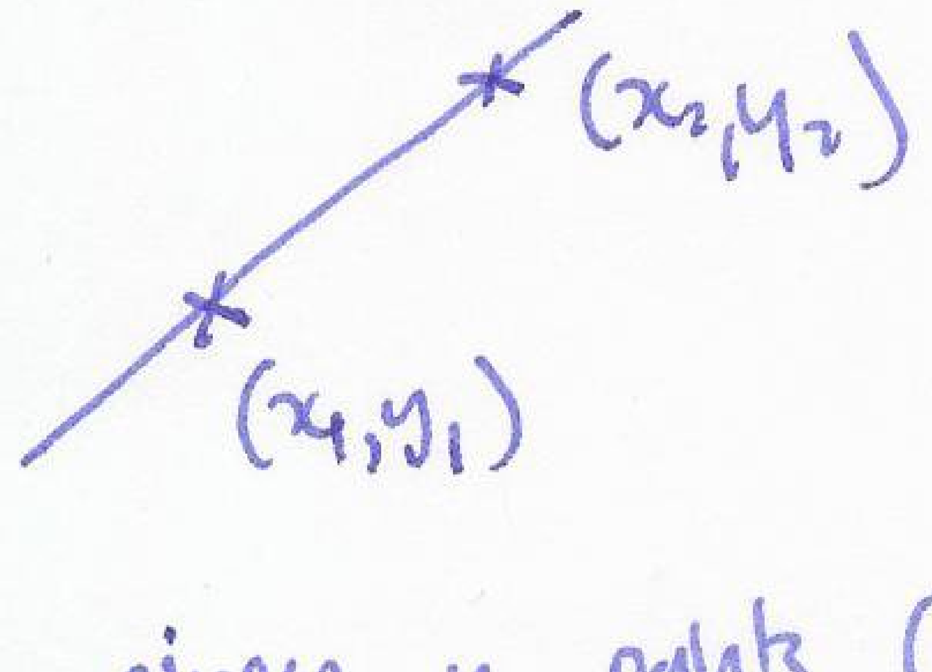
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

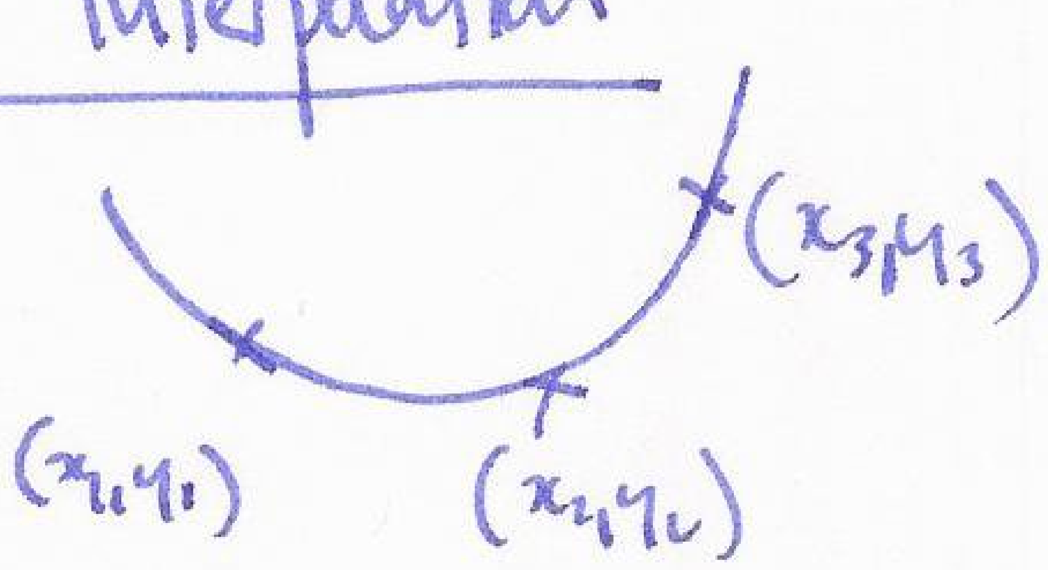
~~Thm~~ **Fact.** A homogeneous system always has at least one solution, i.e., $\underline{x} = \underline{0}$. called the trivial soln.

Thm If $m < n$ (# unknowns > # equations) then a homogeneous system always has a non-trivial solution.

Proof row reduce $\left[\begin{array}{ccc|c} \cdot & \cdots & & 0 \\ & & \ddots & 0 \\ 0 & & & 0 \end{array}\right]$ must be extra columns here $\Rightarrow$ gives a non-trivial solⁿ.

Applications: Polynomial Interpolation





given n points $(x_1, y_1), \ldots, (x_n, y_n)$ in the plane, there is a polynomial of degree $n-1$ which runs through them all.

polynomial: $y = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1 x + a_0$

<u>note</u> unknowns are ~~the~~ a_n!

(x_1, y_1): $\quad a_{n-1}x_1^{n-1} + a_{n-2}x_1^{n-2} + \dots + a_1 x_1 + a_0 = y_1$

$\vdots$

(x_n, y_n): $\quad a_{n-1}x_n^{n-1} + a_{n-2}x_n^{n-2} + \dots + a_1 x_n + a_0 = y_n$

<u>Example</u> $(0,1)$ $(1,1)$ $(2,\cancel{4}\ 3)$.

$$a_2 x^2 + a_1 x + a_0 = y$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 3\cancel{4} \end{array}\right].$$

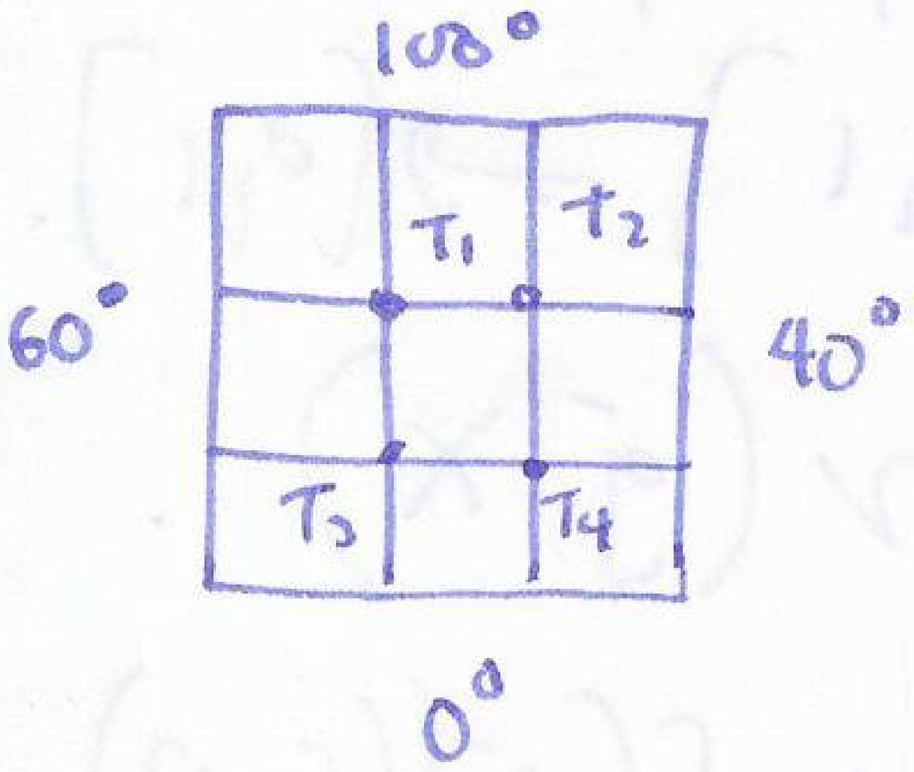
row reduce: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 3\cancel{4} \\ 0 & 0 & 1 & 1 \end{array}\right]$ $\quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & +2 & +3 & +1 \\ 0 & 0 & 1 & 1 \end{array}\right].$

$a_0 = 1 \qquad 2a_1 + 3 = 1 \qquad a_2 - 1 + 1 = 1 \qquad a_2 = 1.$

$a_1 = -1$

$$\boxed{y = x^2 - x + 1}$$

Temperature distribution



temperature at a vertex is the average of the surrounding vertices.

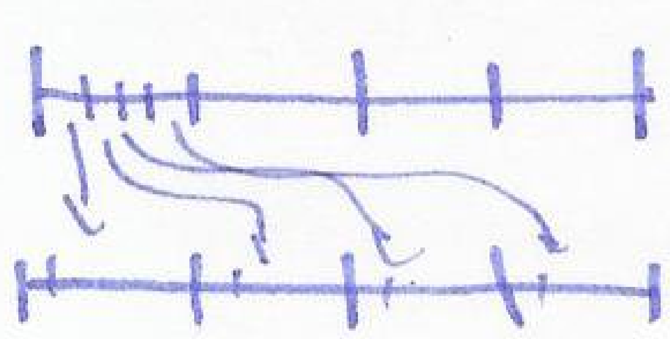
i.e.
$$T_1 = \frac{1}{4}\left(60 + 100 + T_2 + T_3\right).$$
$$T_2 = \frac{1}{4}\left(100 + 40 + T_1 + T_4\right).$$
$$T_3 = \frac{1}{4}\left(0 + 60 + T_1 + T_4\right)$$
$$T_4 = \frac{1}{4}\left(0 + 40 + T_3 + T_2\right)$$

i.e.
$$\begin{aligned} 4T_1 - T_2 - T_3 &= 160 \\ 4T_2 - T_1 - T_4 &= 140 \\ 4T_3 - T_1 - T_4 &= 60 \\ 4T_4 - T_2 - T_3 &= 40 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 160 \\ -1 & 4 & 0 & -1 & 140 \\ -1 & 0 & 4 & -1 & 60 \\ 0 & -1 & -1 & 4 & 40 \end{array}\right]$$

<u>sol$^{\underline{n}}$</u> : $T_1 = 65$, $T_2 = 60$, $T_3 = 40$, $T_4 = 35$.

Markov chains (preview).



pop$^{\underline{n}}$ by wealth (parents). get transition matrix
pop$^{\underline{n}}$ by wealth (children)

$$\begin{bmatrix} p_{11} & & p_{14} \\ & & \\ p_{41} & & p_{44} \end{bmatrix}$$

p_{ij} = prob go from wealth percentile i to j.

§1.7 Inverse of matrix

recall powers A^0, $A^1 = A$, $A^2 = AA$.

Q: what is A^{-1}? would like it to be the inverse.

problem: not every matrix has an inverse.

Defn an $(n \times n)$ matrix is called invertible / nonsingular if there is an $n \times n$ matrix B s.t. $AB = I_n = BA$.

The matrix B is called the inverse, and often written A^{-1}.

If there is no such matrix B, then we say A is invertible or singular.

Example

$$\cancel{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} =}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{no inverse!}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \quad \text{no inverse!}$$

Thm Inverses are unique (if they exist)

Proof spse B, C are inverses for A, i.e. $BA = AC = I_n$

then $B = BI = B(AC) = (BA)C = IC = C$ $\square$.

Thm Useful properties of inverses.

a) if A has an inverse A^{-1}, then A^{-1} has inverse A so

$(A^{-1})^{-1} = A$.

b) if A, B have inverses, then AB has an inverse,

$(AB)^{-1} = B^{-1}A^{-1}$.

c) if A has an inverse, then A^T has an inverse, and

$(A^T)^{-1} = (A^{-1})^T$.

<u>Proof</u> b) check: $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1}$

$= AIA^{-1} = AA^{-1} = I$. $\square$.

c) $AA^{-1} = I$ take transpose: $(AA^{-1})^T = I^T$

$(A^{-1})^T A^T = I$.

so $(A^T)^{-1} = (A^{-1})^T$. $\square$.

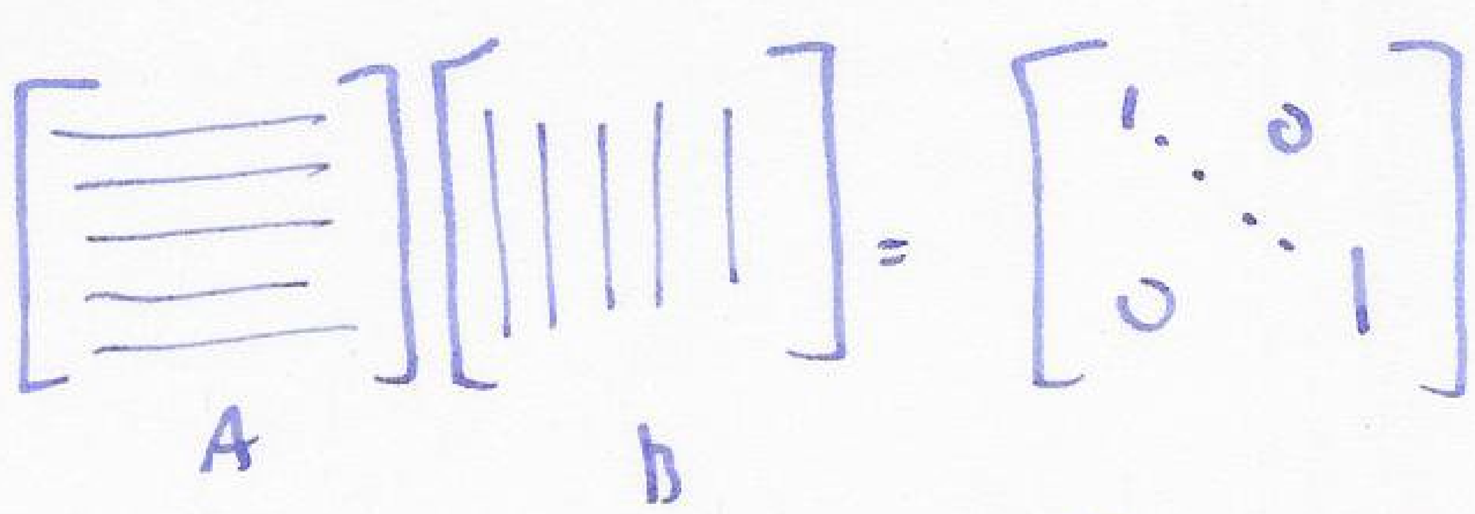
<u>Corollary</u> Let $A_1 A_2 \dots A_n$ be a product of invertible matrices,

then $(A_1 A_2 \dots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \dots A_1^{-1}$.

<u>Finding A^{-1}</u>

i.e. want to find B s.t. $AB = I$.

<u>recall</u>:



consider first column of I $\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ (e_1) $\leftarrow$ values in here only depend on <u>first column</u> of B!

i.e. solve $\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} b_1 \end{bmatrix} = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ this gives first column of A.

2nd column: $A\,\underline{b}_2 = \underbrace{\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}}_{\underline{e}_2}$

corresponding row reductions: $[A\,|\,\underline{e}_1]$ $[A\,|\,\underline{e}_2]$ ... $[A\,|\,\underline{e}_n]$

but can do them all at once $[A\,|\,I]$

Summary procedure:

step 1 form $[A\,|\,I]$ $(n\times 2n)$ matrix.

step 2 compute reduced row echelon form.

step 3 if result is $[I\,|\,B]$ then B is the required inverse

if result is $[C\,|\,B]$ with $C \neq I_n$ (i.e. a row of zeros)

then there is no inverse..

Example $A = \begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 5 & 5 & 1\end{bmatrix}$ $A^{-1} = \begin{bmatrix}13/8 & -1/2 & -1/8\\ -15/8 & 1/2 & 3/8\\ 5/4 & 0 & -1/4\end{bmatrix}$

Example $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$

assume $a \neq 0$.

$ad-bc \neq 0$.

$$\left[\begin{array}{cc|cc}1 & b/a & 1/a & 0\\ 0 & 1 & \frac{-\cancel{a}c}{\cancel{a}(ad-bc)} & \frac{a}{ad-bc}\end{array}\right].$$

$$\left[\begin{array}{cc|cc}a & b & 1 & 0\\ c & d & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{cc|cc}1 & b/a & 1/a & 0\\ c & d & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{cc|cc}1 & 0 & \frac{1}{a} + \frac{bc}{a(ad-bc)} & -\frac{b}{(ad-bc)}\\ 0 & 1 & \frac{-\cancel{a}c}{\cancel{a}(ad-bc)} & \frac{a}{ad-bc}\end{array}\right].$$

$$\left[\begin{array}{cc|cc}1 & b/a & 1/a & 0\\ 0 & \underbrace{d-\frac{bc}{a}}_{=\frac{ad-bc}{a}} & -c/a & 1\end{array}\right]$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} & -b\\ -c & a\end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{ad-bc}{a} & -c/a & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{\cancel{a}c}{a(ad-bc)} & \frac{a}{ad-bc} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}+\frac{b}{a}\frac{c}{ad-bc} & -\frac{b\cancel{a}}{\cancel{a}(ad-bc)} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

$$\frac{1}{a}+\frac{bc}{a(ad-bc)} = \frac{ad-\cancel{bc}+\cancel{bc}}{a(ad-bc)} = \frac{d}{ad-bc} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

<u>useful fact</u> An $n \times n$ matrix is nonsingular iff it is row equivalent to I_n

In particular: if you row-reduce and you get a row of zeros, $\begin{bmatrix} \vdots & & \vdots \\ 0 & 0 \cdots & 0 \end{bmatrix}$ then then matrix is <u>singular</u>.

<u>Thm 1.3</u> If A is an $n \times n$ matrix, then $A\underline{x} = \underline{0}$ has a nontrivial solution iff A is singular.

<u>Proof</u> spose A non-singular, then there is an inverse matrix A^{-1}.

$\Rightarrow$ $A\underline{x} = \underline{0}$ only solution is $\underline{x} = 0$.

$A^{-1}A\underline{x} = A^{-1}\underline{0}$ $\Leftarrow$ row reduce and get row of zeros, gives non-trivial solution.

$I_n\underline{x} = \underline{0}$ □