# §1.6 Solving linear equations

some linear equations are easy to solve:

$$\begin{aligned} x_1 + \qquad\qquad 2x_4 &= 4 \\ x_2 + \qquad -x_4 &= -5 \\ x_3 + 3x_4 &= 6 \end{aligned} \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & 3 & 6 \end{array}\right]$$

solution:

$$\begin{aligned} x_3 &= 6 - 3x_4 \\ x_2 &= -5 + x_4 \\ x_1 &= 4 - 2x_4 \end{aligned}$$

$A\underline{x} = \underline{b}$

$$\underline{x} = \begin{bmatrix} 4-2x_4 \\ -5+x_4 \\ 6-3x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_4 \\ x_4 \\ -3x_4 \\ x_4 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \\ 6 \\ 0 \end{bmatrix}.$$

$$\underline{x} = x_4 \begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \\ 6 \\ 0 \end{bmatrix}$$

<u>Def<sup></sup></u>

**Def^n** An $(m \times n)$ matrix $A$ is in <u>reduced row echelon form</u> if

a) all zero rows (if any) appear at the bottom of the matrix

b) the first non zero entry from the left in any (non-zero) row is 1, called the <u>leading one</u> of the row.

c) for each non-zero row, the leading one appears to the right of the leading one in the rows above.

d) if a column contains a leading one, then all other entries are zero.

If you just satisfy a) b) c) but <u>not</u> d) you are in row-echelon form.

system of linear equations $\leftrightarrow$ matrix, not necessarily in row echelon form.

how to get ~~the~~ a matrix into row-echelon form:

Def$^{\underline{n}}$ An elementary row operation on an $m \times n$ matrix $A$ is one of the following operations:

a) swap any two rows of $A$

b) multiply a row by a number $c \neq 0$.

c) add a multiple of one row to any other row.

Example
$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix} \overset{\text{swap } 1,2}{\rightsquigarrow} \begin{bmatrix} 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 3 & 3 & 6 & -9 \end{bmatrix}$$

$$\underset{\text{multiply row 3 by } \frac{1}{3}}{\rightsquigarrow} \begin{bmatrix} 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & -3 \end{bmatrix} \underset{\text{add row 2 to row 1}}{\rightsquigarrow} \begin{bmatrix} 2 & 3 & \cancel{0}\,1 & \cancel{-2}\,0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & -3 \end{bmatrix}.$$

Def$^{\underline{n}}$ An $(m \times n)$ matrix $A$ is row equivalent to an $(m \times n)$ matrix $B$ if $B$ may be obtained from $A$ by a finite sequence of row operations.

Th$^{\underline{m}}$ Every $(m \times n)$ matrix is row equivalent to a matrix in (reduced) row echelon form.

**Example**

$$\text{pivot} \rightarrow \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

**Procedure** **step ①** find first non-zero column, this is called the pivot column

**step ②** find the first non-zero entry in the column. This is the pivot

**step ③** swap the ~~top~~ first row with the pivot row, so the pivot is in the first row.

$$\begin{bmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

**step ④** multiply the first row by 1/pivot ~~to~~ so the pivot becomes a leading one.

$$\begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

**step ⑤** add multiples of row 1 to any row with a non-zero entry in the pivot column, so the pivot column contains a single one in the first row.

$$\begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \quad B$$

now let $B$ be the $(m-1)\times n$ matrix formed by ignoring first row.
apply same procedure to $B$.

pivot

$$\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & -4 & 1 \\ -2 & -1 & 7 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & -4 & 1 \\ 0 & 2 & 3 & 4 \\ -2 & -1 & 7 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3/2 & -2 & 1/2 \\ 0 & 2 & 3 & 4 \\ -2 & -1 & 7 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & -2 & 1/2 \\ 0 & \boxed{\begin{matrix} 2 & 3 & 4 \\ 2 & 3 & 4 \end{matrix}} \\ 0 & & & \end{bmatrix}$$

pivot

$$\begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3/2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3/2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

~~final answer~~ ?
now get:

$$\begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 1 & 3/2 & -2 & 1/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

this in row-echelon form but is not reduced.

to get reduced row echelon form, add multiples of columns with leading ones to those above:

$$\begin{bmatrix} 1 & 0 & -4 & 3 & 3/2 \\ 0 & 1 & 3/2 & -2 & 1/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 9 & 19/2 \\ 0 & 1 & 0 & -17/4 & -5/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

↑ reduced row echelon form.

<u>Thm</u> Let $A\underline{x} = \underline{b}$ and $C\underline{x} = \underline{d}$ be two systems of linear equations. If $[A|\underline{b}]$ and $[C|\underline{d}]$ are row equivalent, then both linear systems have exactly the same set of solutions.

<u>Proof</u> need to check each operation.

① <u>swaps</u>

$$A \cdot \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n = b_r \\ \vdots \\ a_{s1}x_1 + a_{s2}x_2 + \cdots + a_{sn}x_n = b_s \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$A' \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{s1}x_1 + \cdots + a_{sn}x_n = b_s \\ \vdots \\ a_{r1}x_1 + \cdots + a_{rn}x_n = b_r \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

if $x_1, \dots x_n$ satisfies $A$, then it also satisfies $A'$

if $x_1, \dots x_n$ satisfies $A'$, then it also satisfies $A$.

② multiply a row by a non-zero number.

$$A \left[ \begin{array}{l} \vdots \\ a_{r1}x_1 + \cdots + a_{rn}x_n = b_r \\ \vdots \end{array} \right. \qquad A' \left[ \begin{array}{l} \vdots \\ ca_{r1}x_1 + \cdots + ca_{rn}x_n = cb_r \\ \vdots \end{array} \right.$$

what goes wrong if $c = 0$?