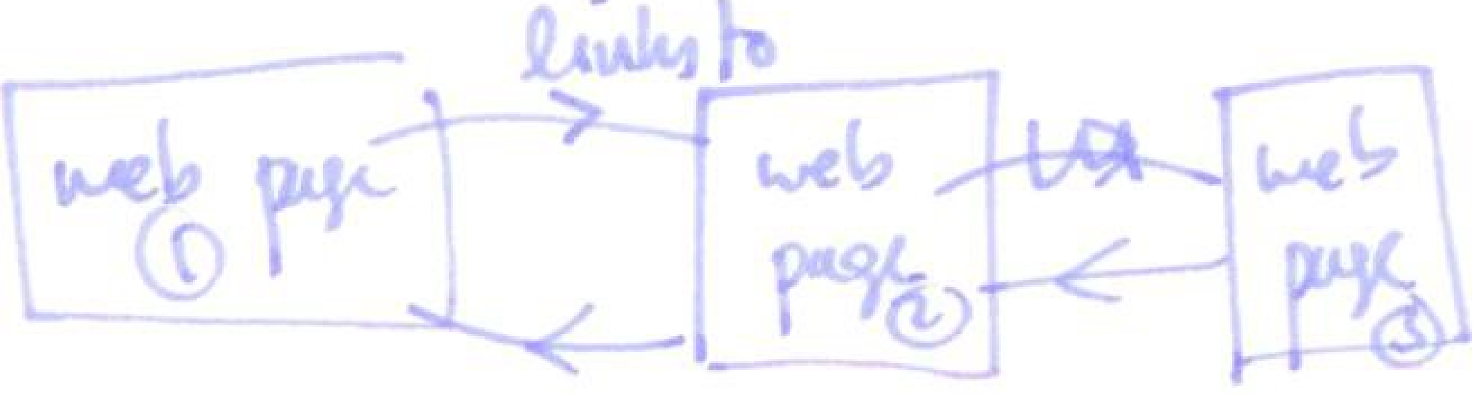
~~Ex~~ Example : Google's matrix

webwork due Sun midnight.



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$a_{ij} = 1$ if web page i links to j

$= 0$ otherwise.

Defn two matrices A, B are equal if same number of rows and cols and $a_{ij} = b_{ij}$ for all i, j.

Matrix addition A, B $m \times n$ matrices, then ~~$(A+B)_{ij} = A_{ij} + B_{ij}$~~.

$C = A + B$ where $c_{ij} = a_{ij} + b_{ij}$

example $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 \\ 4 & 4 & 6 \end{bmatrix}$.

Scalar multiplication A $m \times n$ matrix r number

$B = rA$ then $b_{ij} = r a_{ij}$.

example $2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$.

Linear combinations of matrices

$A_1, A_2, \dots A_k$ $m \times n$ matrices $c_1, c_2, \dots c_k$ numbers.

then $c_1 A_1 + c_2 A_2 + \dots + c_k A_k$ is an $m \times n$ matrix which is ~~called~~ a linear combination of $A_1, \dots, A_k$.

Transpose of matrices

A $m \times n$ matrix then the transpose A^T is an $n \times m$ matrix, where $a^T_{ij} = a_{ji}$

example
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

§1.3 Dot products and matrix multiplication

recall the dot product of two vectors $\underline{a}, \underline{b} \in \mathbb{R}^n$ is

$$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + \cdots a_nb_n \qquad \underline{a} = [a_1, \ldots a_n] \quad \underline{b} = [b_1, \ldots b_n]$$

important: dimension of $\underline{a}, \underline{b}$ must be the same.

Matrix multiplication

Let A be a $p \times q$ matrix
B $q \times r$ matrix

we define the matrix product AB to be the $p \times r$ matrix whose ij-th element is the dot product of the ith row of A with the j-th column of B.

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 11 & 9 \end{bmatrix}$$

$(2\times 3) \quad (3\times 2) \quad (2\times 2)$

important:

$$\underset{p\times q}{A} \quad \underset{q\times r}{B}$$

result is a $p\times r$ matrix.

must be the same.

general notation:

$$\underset{p\times q}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & & & \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix}}} \underset{q\times r}{\overset{B}{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & & & \\ b_{q1} & b_{q2} & \cdots & b_{qr} \end{bmatrix}}} = \overset{C}{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pr} \end{bmatrix}}$$

where

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{iq} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{qj} \end{bmatrix}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{iq}b_{qj} = \sum_k a_{ik}b_{kj}$$

warning ① always check dimensions work.

in Example

$$AB = \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 4 & 9 \end{bmatrix}$$

but BA is ~~not defined~~. (3x3)

② even if BA is defined $AB \neq BA$ in general.

example

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}} = \begin{bmatrix} 4 & 1 \\ 11 & 1 \end{bmatrix}$$

~~AB~~.

$$\overset{B}{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \begin{bmatrix} -2 & -2 \\ 5 & 8 \end{bmatrix}$$

$AB \neq BA$.

order matters!

observation spose $\underline{u}$ is a $1 \times n$ matrix

$\underline{v}$ is a $n \times 1$ matrix

then $\underset{(1\times n)(n\times 1)}{\underline{u} \cdot \underline{v}} = \underline{u}\,\underline{v}$

$\underline{u} \cdot \underline{u} = \underset{(1\times n)(n\times 1)}{\underline{u} \cdot \underline{u}^T}$

$\underline{v} \cdot \underline{v} = \underline{v}^T \underline{v}$

$\underline{v} \cdot \underline{u} = \underline{v}^T \underline{u}^T$

matrix times vector as a sum

$$A\underline{c} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \text{row}_1(A)\cdot\underline{c} \\ \text{row}_2(A)\cdot\underline{c} \\ \vdots \\ \text{row}_n(A)\cdot\underline{c} \end{bmatrix}$$

~~$(m\times m)$~~
$(m\times n)(n\times 1)$

$$= \begin{bmatrix} a_{11}c_1 + a_{12}c_2 + \cdots a_{1n}c_n \\ \vdots \\ \vdots \\ a_{m1}c_1 + a_{m2}c_2 + \cdots a_{mn}c_n \end{bmatrix}$$

$$= c_1\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= c_1\,\text{col}_1(A) + c_2\,\text{col}_2(A) + \cdots + c_n\,\text{col}_n(A)$$

~~Linear~~ Systems of linear equations

recall: m equations in n-unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \longleftrightarrow \quad A\underline{x} = \underline{b}$$

$(m\times n)(n\times 1) = (m\times 1)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Summation notation

recall $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

note: $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k$ etc...

useful properties: $\sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$ $\qquad$ $\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}$.

we can write matrix products in this notation:

dot product: $\underline{a} = [a_1, \ldots a_n]$ $\quad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

then $\underline{a} \cdot \underline{b} = \sum_{i=1}^{n} a_i b_i$

if $AB = C$ then $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

§1.4 Matrix operations

Matrix addition A, B, C, D $m\times n$ matrices.

~~$A+B = B+A$~~ recall $A+B$ is an $m\times n$ matrix

$$[A+B]_{ij} = a_{ij} + b_{ij}$$

properties:

- $A+B = B+A$
- $A+(B+C) = (A+B)+C$
- there is a unique $m\times n$ matrix O ($[O]_{ij} = 0$).

  s.t. $A+O = A$

  O is called the zero matrix.
- given A there is a unique $m\times n$ matrix D s.t.

  $A+D=O$

  we shall ~~call D~~ write $-A$ for D

  $$[-A]_{ij} = -a_{ij}.$$

Proof

$A+B = B+A$: $a_{ij} + b_{ij} = b_{ij} + a_{ij}$.

$A+(B+C) = (A+B)+C$: $a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$

$A+O = A$: $a_{ij} + 0 = a_{ij}$.

$A+(-A) = O$: $a_{ij} + (-a_{ij}) = 0$.

Matrix multiplication

Assume A, B, C have the correct dimensions so the following multiplications are defined.

- $(AB)C = A(BC)$
- $A(B+C) = AB + AC$
- $(A+B)C = AC + BC$

] but $A(B+C) \neq (B+C)A$ etc.!

Proof $A(B+C) = AB + AC$

$$A(B+C)_{ij} = \text{row}_i(A) \cdot \text{col}_j(B+C) = \text{row}_i(A)\big(\text{col}_j(B) + \text{col}_j(C)\big).$$

$$= \text{row}_i(A)\,\text{col}_j(B) + \text{row}_i(A)\,\text{col}_j(C).$$

$$= AB_{ij} + AC_{ij}.$$

$(AB)C = A(BC)$

$$[(AB)C]_{ij} = \sum_{k=1}^{n} [AB]_{ik} c_{kj} = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{il} b_{lk} c_{kj}$$

$$= \sum_{l=1}^{m} \sum_{k=1}^{n} a_{il} b_{lk} c_{kj} = \sum_{l=1}^{m} a_{il} \sum_{k=1}^{n} b_{lk} c_{kj}$$

$$= \sum_{l=1}^{m} a_{il} [BC]_{lj} = [A(BC)]_{ij}.$$