Math 338 Linear Algebra

Joseph Maher joseph.maher@csi.cuny.edu

web page: http://www.math.okstate.edu/~maher

office 1S-222 office hours: Tu 4:30-6:30 Fr 4:30-5:30

- math tutoring: 1S-214
- students with disabilities

Text: Kolman+Hill, Introductory linear algebra.

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§1.1 Linear equations ~~and matrices~~

linear equation: $ax+by=c$ not linear: $x^2+y^2=1$

$xy=1$.

$\sin(x)=e^x$.

can have: many variables $ax+by+cz=d$.

$$a_1x_1+a_2x_2+a_3x_3+a_4x_4=b$$

$$a_1x_1+a_2x_2+\dots+a_nx_n=b.$$

many equations: $\left.\begin{array}{l} ax+by=c \\ dx+ey=f \end{array}\right\}$ pair of equations in two unknowns

solution: pair of number $x=s, y=t$ s.t. $\begin{array}{l} as+bt=c \\ ds+et=f. \end{array}$

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=b_m \end{array}\right\} \text{system of } m \text{ equations in } n \text{ unknowns.}$$

<u>solutions</u> a solution is a set of numbers $x_1 = s_1, x_2 = s_2 \dots x_n = s_n$

s.t. $a_{11}s_1 + a_{12}s_2 + \dots + a_{1n}s_n = b_1$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1}s_1 + a_{m2}s_2 + \dots + a_{mn}s_n = b_m .$$

<u>Q</u>:
- when does a system of equations have a solution?
- how can we find a solution?
- how can we find <u>all</u> possible solutions?
- how can we find all solutions in a systematic manner?

<u>Examples</u>

① $\left.\begin{array}{ll} x+y = 1 & ① \\ x-y = 2 & ② \end{array}\right\}$ ① : $x+y = 1$
② − ① : $-2y = 1$ $y = -\frac{1}{2}$
$x = \frac{3}{2}$

solution: $(x,y) = \left(\frac{3}{2}, -\frac{1}{2}\right)$

② $\left.\begin{array}{ll} x+y = 1 & ① \\ -2x-2y = 3 & ② \end{array}\right\}$ ① : $x+y=1$
② + 2① : $0x+0y = 5$ ⨳ no solutions!.

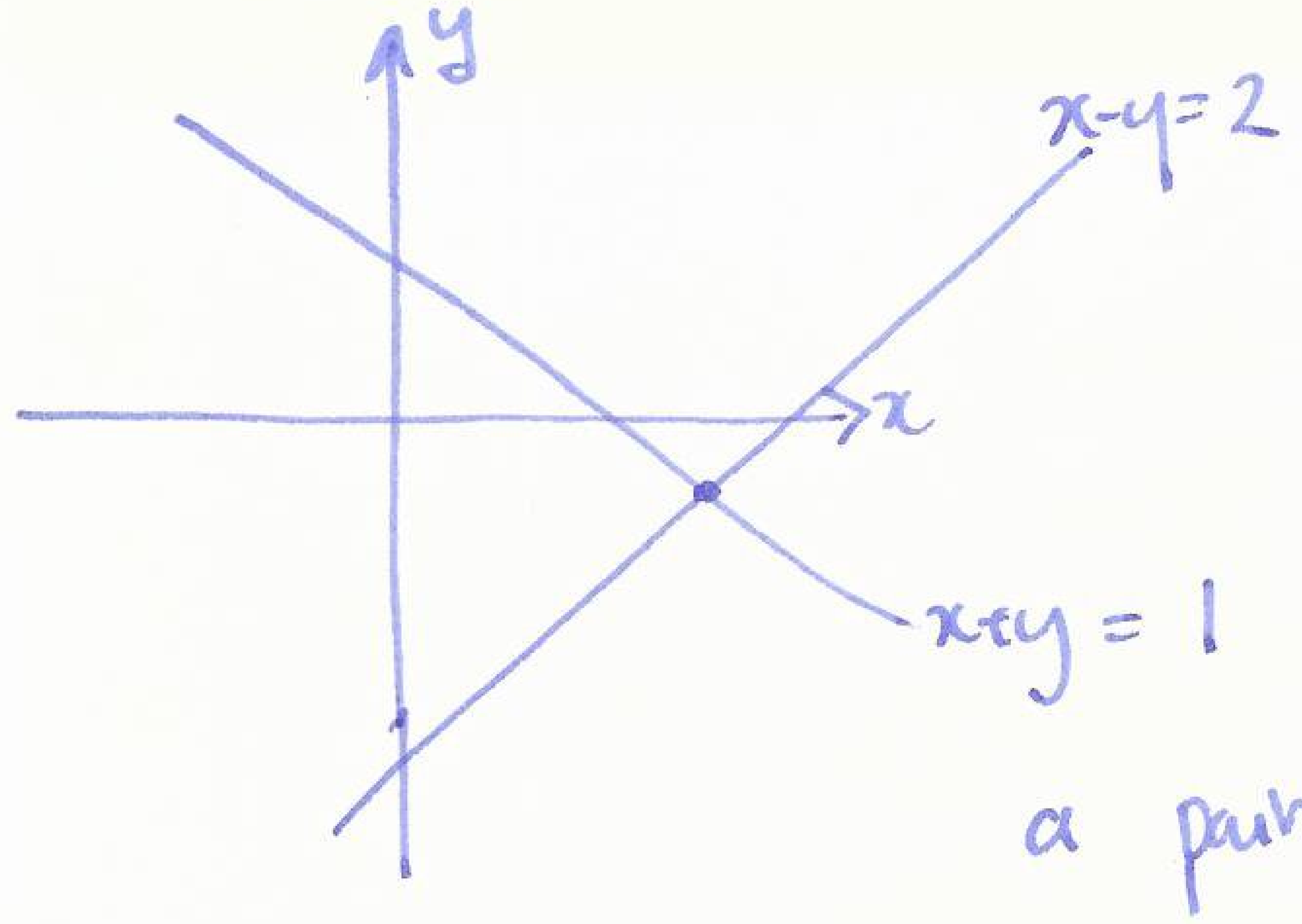
③ $\left.\begin{array}{ll} x+y = 1 & ① \\ -2x-2y = -2 & ② \end{array}\right\}$ $\left.\begin{array}{ll} ① : & x+y=1 \\ ②+2① : & 0x+0y = 0 \end{array}\right\}$ in fact infinitely many solutions:

$(x,y) = (t, 1-t)$

check: $t + (1-t) = 1 \iff 1 = 1$ ✓.

$-2(t) - 2(1-t) = -2t - 2 + 2t = -2$ ✓.

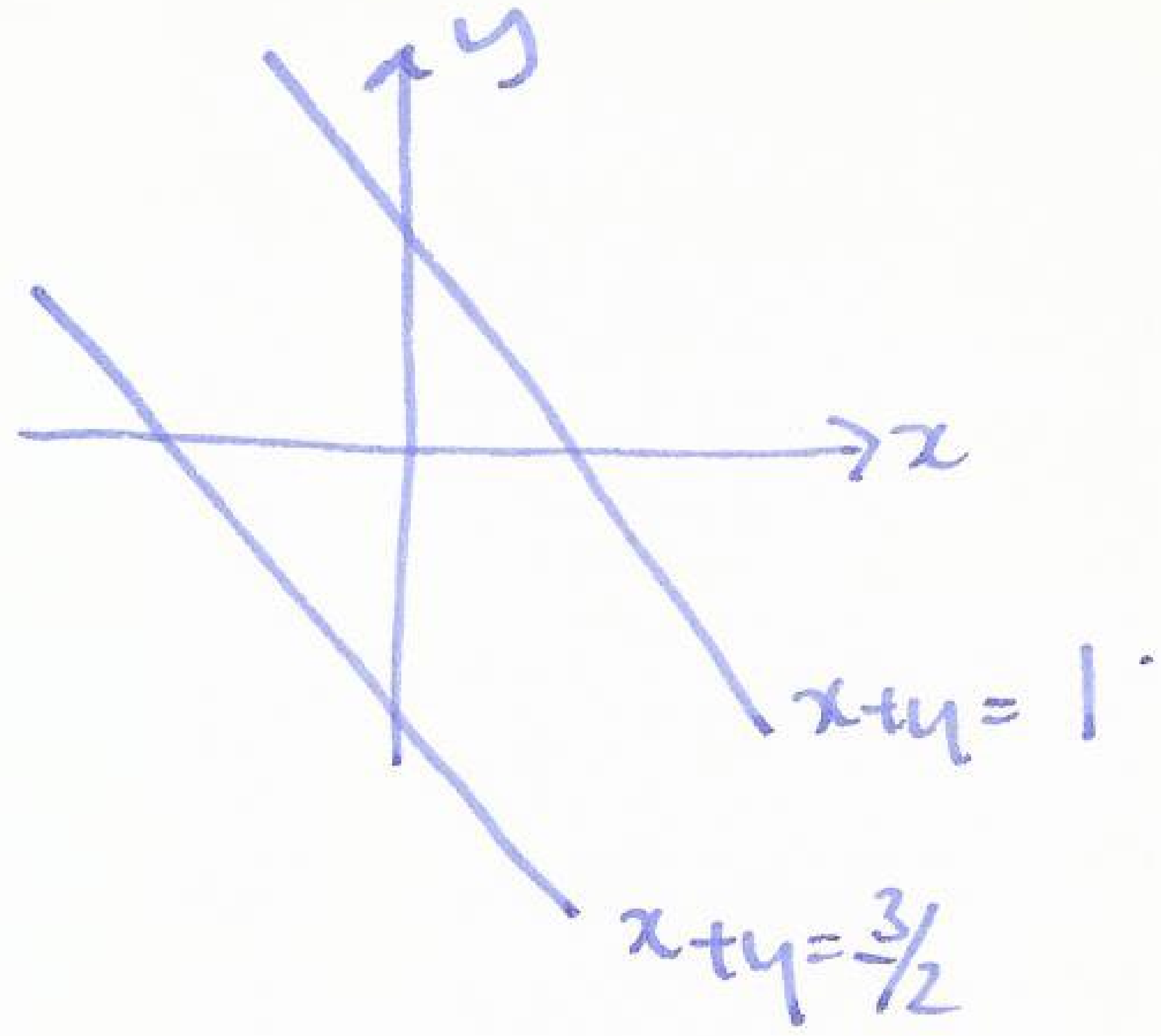
1



a linear equation in two variables x,y describes a line in the xy-plane.

a pair of non-parallel lines intersect in a single point : unique solution

2

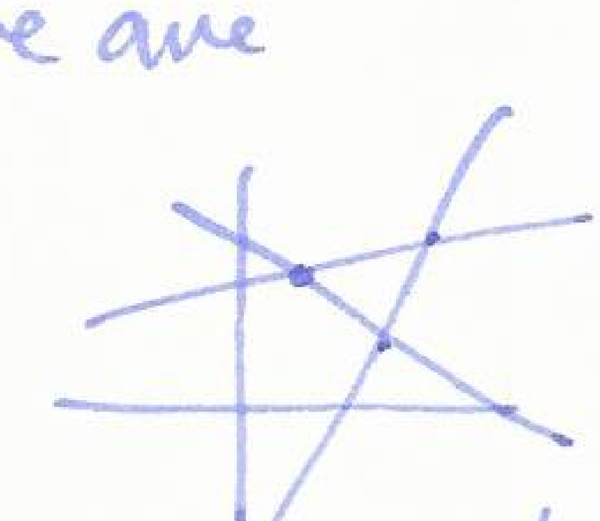


a pair of parallel lines do not intersect : no solutions.

3

same line twice : any point on the line is a solution, infinitely many solutions (and we can describe them!).

Q: what if there are three lines?

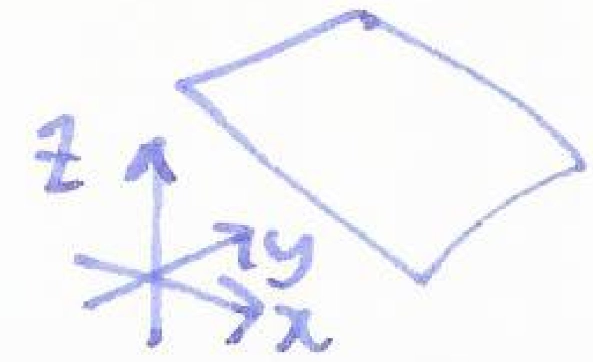


no solution here!

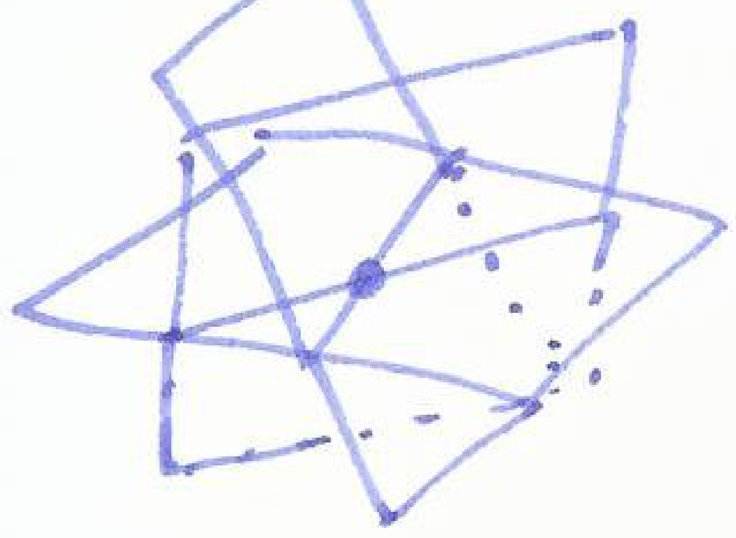
__Example__

$$x + 2y + 3z = 6$$
$$2x - 3y + 2z = 14$$
$$3x + y - z = -2$$

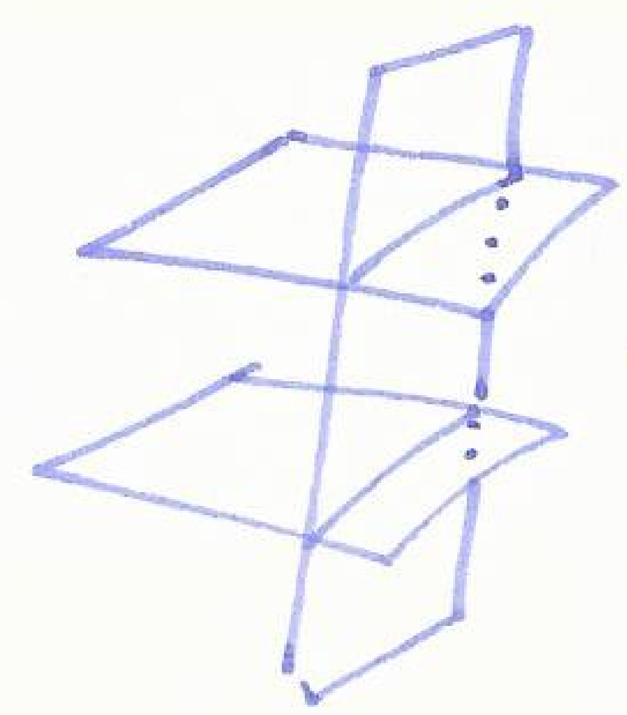
__Q__: what can happen? a linear equation in 3 variables describes a __plane__



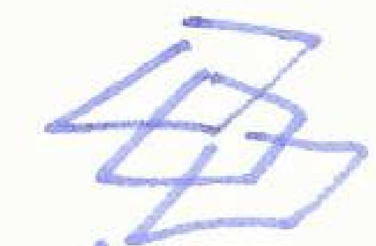
how can three planes intersect?



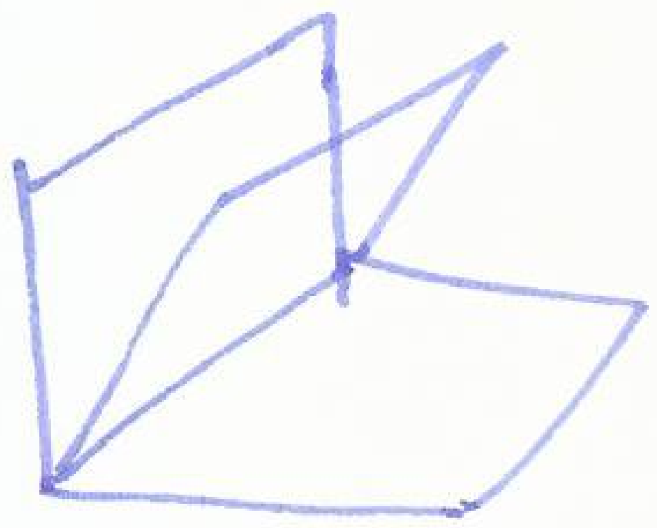
← unique solutions



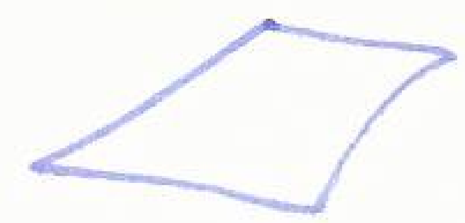
← no solutions.



also no sol^{ns}.

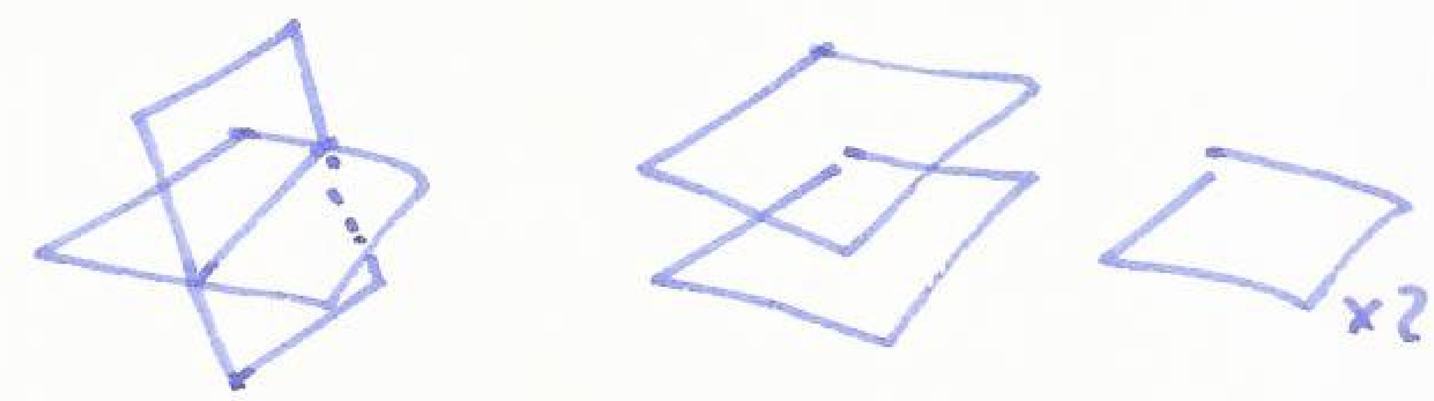


← infinitely many solutions.

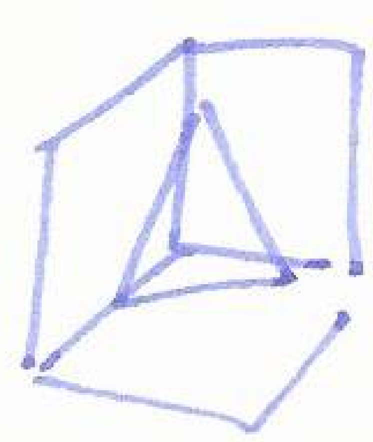


same plane 3 times
- also no solutions.

Q: what if there are only two planes?



- what if there are four planes



← no solutions here

Example (solve by elimination).

$$\begin{aligned} x + 2y + 3z &= 6 \quad ① \\ 2x - 3y + 2z &= 14 \quad ② \\ 3x + y - z &= -2 \quad ③ \end{aligned}$$

$$\begin{aligned} ①: \quad & x + 2y + 3z = 6 \\ ② - 2①: \quad & 0x - 7y - 4z = 2 \\ ③ - 3①: \quad & 0x - 5y - 10z = -20 \end{aligned}$$

$$\begin{aligned} x + 2y + 3z &= 6 \quad ① \\ y + 2z &= 4 \quad ② \\ -7y - 4z &= 2 \quad ③ \end{aligned}$$

$$\begin{aligned} ①: \quad & x + 2y + 3z = 6 \\ ②: \quad & y + 2z = 4 \\ ③ + 7②: \quad & 0y + 10z = 30 \end{aligned}$$

$z = 3$, $\quad y + 6 = 4$, $y = -2$, $\quad x + 2(-2) + 3.3 = 6$

$x - 4 + 9 = 6 \quad x = 1.$

Elimination rules

1. swap two equations
2. multiply an equation by a non-zero constant
3. add a multiple of one equation to another.

observation: in elimination we never changed the order of the variables x, y, z, so we could have done the elimination just writing down the coefficients, i.e.

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{bmatrix}$$

etc.

§1.2 Matrices

Defn An $m \times n$ matrix A is a rectangular array of mn numbers arranged in m horizontal rows and n columns.

(rows) x (columns).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

example

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix}$$

3x4 matrix.

2nd row.

3rd column.

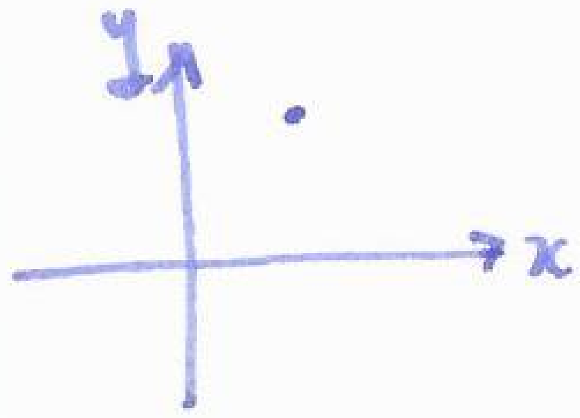
in general: $\begin{bmatrix} a_{i1} \dots & a_{in} \end{bmatrix}$ ith row of A

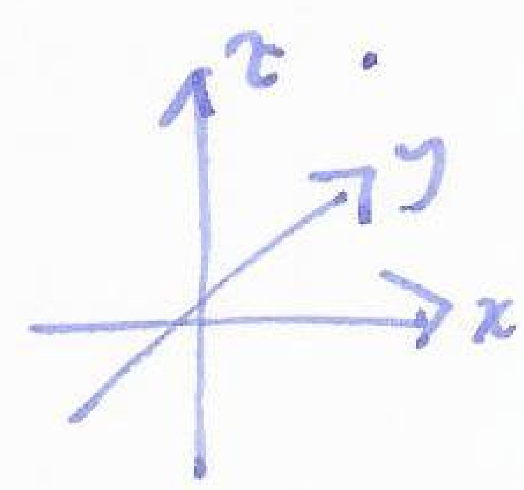
$\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$ j-th column of A.

$[a_{ij}]$ i,j-th element of A.

<u>Defn</u> an $n \times 1$ matrix or an $1 \times n$ matrix is also known as a <u>vector</u>. (or an n-vector). The set of all n-vectors is called $\mathbb{R}^n$.

<u>Examples</u> $[1,2] \in \mathbb{R}^2$ $[1,2,3] \in \mathbb{R}^3$





<u>Special matrices</u>

- 0-vector $[0\ 0 \dots 0]$ or $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ sometimes written $\underline{0}$
- square matrix: #rows = #cols. $[1]$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ ....
- diagonal matrix: $[1]$ $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ...

  $a_{ij} = 0$ if $i \neq j$
- scalar matrix: diagonal elements all the same: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.