Linear Algebra Spring 10 Midterm 3

Name: Solutions

- You may use a calculator, but no notes.

1	15	
2	15	
3	20	
4	25	
5	20	
6	20	
	115	

(1) (15 points)

(a) Write down an othonormal basis for $\mathbb{R}^3$.

(b) Write down a basis for $\mathbb{R}^3$ which is orthogonal, but not orthonormal.

(c) Write down a basis for $\mathbb{R}^3$ which is not orthogonal.

(d) Write down a spanning set for $\mathbb{R}^3$ which is not a basis.

(e) Write down a set of linearly independent vectors in $\mathbb{R}^3$ which is not a basis.

a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(2) (15 points)
Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Use the Gram-Schmidt process to find an orthogonal basis.
(b) Now find an othonormal basis.

a) set $\underline{w_1} = \underline{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\underline{w_2} = \underline{v_2} - \lambda \underline{w_1}$ where $\lambda = \dfrac{\underline{w_1} \cdot \underline{v_2}}{\underline{w_1} \cdot \underline{w_1}} = \dfrac{6}{6} = 1$

$\underline{w_2} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\underline{w_3} = \underline{v_3} - \lambda_1 \underline{w_1} - \lambda_2 \underline{w_2}$ where $\lambda_1 = \dfrac{\underline{w_1} \cdot \underline{v_3}}{\underline{w_1} \cdot \underline{w_1}} = \dfrac{6}{6} = 1$

$\lambda_2 = \dfrac{\underline{w_2} \cdot \underline{v_3}}{\underline{w_2} \cdot \underline{w_2}} = \dfrac{2}{2} = 1$

$\underline{w_3} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ orthogonal basis.

b) $\left\{ \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \right\}$ orthonormal basis.

(3) (20 points) $\underline{v_1}$ $\underline{v_2}$ $\underline{v_3}$.

Let $V = \text{Span}\{(1,2,1,-1),(2,1,-1,4),(1,-1,-2,-5)\}$ in $\mathbb{R}^4$.

(a) What is the dimension of V?

(b) Find a basis for $V^{\perp}$.

a)
$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 1 & -1 & 4 \\ 1 & -1 & -2 & -5 \end{bmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

span rows of A = span rows of rref(A) so $\dim V = 3$.

b) solve $\underline{x} \cdot \underline{v}_i = 0$ i.e. solve $A\underline{x} = \underline{0}$ where $A = \begin{bmatrix} \underline{v_1} \\ \underline{v_2} \\ \underline{v_3} \end{bmatrix}$

$x_4 = 0$

$x_3 = t$

$x_2 + x_3 = 0$

$x_1 - x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $V^{\perp}$.

(4) (25 points)

$$A = \begin{bmatrix} 7 & 4 \\ -8 & -5 \end{bmatrix}$$

(a) Find the eigenvalues of A.
(b) Find the eigenvectors for A.
(c) Diagonalize A, i.e. find matrices P and D such that $P^{-1}AP = D$.

a) $(7-\lambda)(-5-\lambda) + 32 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$

$\lambda = -1, 3.$

b) $\lambda = -1$: $\begin{bmatrix} 8 & 4 \\ -8 & -4 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 3$: $\begin{bmatrix} 4 & 4 \\ -8 & -8 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

c) $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

check $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -8 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ ✓

(5) (20 points)
Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L(x,y) = (2x+y, x-2y)$. Let

$$S = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^2$, and let T be the standard basis for $\mathbb{R}^2$.
(a) Find the matrix for L with respect to T.
(b) Find the matrix for the change of basis from S to T.
(c) Find the matrix for L with respect to S.

a) $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

b) $P = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

c)
$$\begin{tikzcd} \mathbb{R}^2_T \arrow[r, "{\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}}", "A"'] & \mathbb{R}^2_T \\ \mathbb{R}^2_S \arrow[u, "P"] \arrow[r] & \mathbb{R}^2_S \arrow[u, "P"'] \end{tikzcd}$$

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -1 & 3 \end{bmatrix}$$

(6) (20 points)

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

(a) Find the eigenvalues for A.
(b) Find the eigenvectors for A.
(c) Can you diagonalize A? Explain.

a) $(3-\lambda)(1-\lambda)+1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$

$\lambda = 2, 2$.

b) $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvector.

c) No. not enough eigenvectors.