Linear Algebra Spring 10 Sample midterm 3

(1) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Use the Gram-Schmidt process to find an orthogonal basis.
(b) Now find an othonormal basis.

(2) (a) Let $V = \text{Span}\{(-1, 1, 2), (2, 4, -2)\}$ in $\mathbb{R}^3$. Find a basis for $V^\perp$.
(b) If $A^2 = A^T$, what are the possible eigenvalues of A? Justify.

(3)

$$A = \begin{bmatrix} 3 & 1 \\ -5 & -3 \end{bmatrix}$$

(a) Find the eigenvalues of A.
(b) Diagonalize A, i.e. find matrices P and D such that $P^{-1}AP = D$.
(c) Find an exact formula for A^k.
(d) What are the eigenvalues for A^k?

(4)

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 & 6 \\ 2 & 1 & 1 & 1 & 3 \\ -1 & 4 & 10 & 1 & 12 \end{bmatrix}$$

(a) Find the rank and nullity of A. Justify.
(b) Find a basis for the orthogonal complement of the null space.

(5) Let P_4 be the vector space of polynomial of degree at most 4.
(a) Write down a basis for P_4.
(b) Is the set $\{t^4 + 1, t^3 + t, t^2\}$ linearly independent. Justify.
(c) What is the orthogonal complement to P^3 in P^4?

(6) Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L(x, y) = (x - 2y, x + 2y)$. Let

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^2$, and let T be the standard basis for $\mathbb{R}^2$.
(a) Find the matrix for L with respect to T.
(b) Find the matrix for L with respect to S.
(c) What is the rank and nullity of L?

(7) (a) Write down a matrix representing a rotation of θ anti-clockwise about the origin in $\mathbb{R}^2$.
(b) What are the eignvalues and eigenvectors of the matrix?

(8) (a) Explain why every 3×3 matrix has a real eigenvalue.
(b) What does this say about rotations in $\mathbb{R}^3$?
(c) Write down a real 4×4 matrix all of whose eigenvalues are complex.
(d) What does this say about rotations in $\mathbb{R}^4$?