Linear Algebra Sample Midterm 3 Solutions

Q1 choose $\underline{w_1} = \underline{v_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

choose $\underline{w_2} = \underline{v_2} - \lambda \underline{v_1}$ where $\lambda = \dfrac{w_1 . v_2}{\| v_1 . w_1 \|} = \dfrac{3}{6} = \dfrac{1}{2}$

$$\underline{w_2} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

choose $\underline{w_3} = \underline{v_3} - \lambda_1 \underline{w_1} - \lambda_2 \underline{w_2}$

where $\lambda_1 = \dfrac{\underline{v_3} . \underline{w_1}}{\underline{w_1} . \underline{w_1}} = \dfrac{1}{6}$ $\lambda_2 = \dfrac{\underline{v_3} . \underline{w_2}}{\underline{w_2} . \underline{w_2}} = \dfrac{-1/2}{1/2} = -1$

so $$\underline{w_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

b) normalize : $$\left\{ \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}.$$

Q2 a) find $\underline{v}$ s.t. $\underline{v} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = 0$ and $\underline{v} \cdot \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = 0$

i.e. solve $\left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 2 & 4 & -2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 6 & 2 & 0 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 7/3 & 0 \\ 0 & 1 & 1/3 & 0 \end{array}\right]$ $\quad \underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $z = t$

$y = -\frac{1}{3}t$

$z = -\frac{7}{3}t$.

so sol^ns are : $t \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix}$

so $\left\{ \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix} \right\}$ is a basis for $V^{\perp}$.

b) $A = P^{-1} D P$ D upper triangular

$A^2 = P^{-1} D^2 P$

$A^T = P^T D^T P^{T^{-1}}$ D^T lower triangular.

$A^2 \underline{v} = \lambda \underline{v} = A^T \underline{v}$ if $\underline{v}$ eigenvector.

$\Rightarrow D^2 = (PP^T) D^T (PP^T)^{-1}$ similar so if eigenvalues

for A $\{\lambda_1 \dots, \lambda_n\}$ then $\{\lambda_1^2, \dots, \lambda_n^2\}$ also eigenvalues.

$\Rightarrow$ all eigenvalues roots of unity.

Q3

a) $A = \begin{bmatrix} 3 & 1 \\ -5 & -3 \end{bmatrix}$ $\quad \det(A - \lambda I) = (3-\lambda)(-3-\lambda) + 5$

$\lambda^2 - 4 = 0 \quad \lambda = \pm 2.$

b) $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -5 & -5 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is eigenvector corresponding to $+2$.

$\left[\begin{array}{cc|c} 5 & 1 & 0 \\ -5 & 1 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ is eigenvector corresponding to -2.

So $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = P^{-1}AP$ where $P = \begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix}$

c) $D^k = \begin{bmatrix} 2^k & 0 \\ 0 & (-2)^k \end{bmatrix}$

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix} \cdot -\frac{1}{4}$$

$$= -\frac{1}{4} \begin{bmatrix} 2^k & (-2)^k \\ -2^k & -5(-2)^k \end{bmatrix} \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -5 \cdot 2^k + (-2)^k & -2^k + (-2)^k \\ +5 \cdot 2^k - 5(-2)^k & 2^k - 5(-2)^k \end{bmatrix}$$

d) 2^k and $(-2)^k$.

Q4 a) row reduce :
$$\begin{bmatrix} 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 1 & 7/3 & 1/3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank$(A) = 2$ (row rank = rank).

nullity $(A) = 3$ (rank nullity formula : rank(A) + nullity(A) = #cols)

2 + nullity$(A) = 5$.

b) orthogonal complement of null space = row space

has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2/3 \\ 1/3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7/3 \\ 1/3 \\ 3 \end{bmatrix} \right\}$.

Q5 a) $\{1, t, t^2, t^3, t^4\}$.

b) yes. suppose $c_1(t^4+1) + c_2(t^3+t) + c_3(t^2) = 0$

$c_1 t^4 + c_2 t^3 + c_3 t^2 + c_2 t + c_1 = 0_{\text{polynomial}}$.

$\Rightarrow$ all $c_i = 0$.

c) which inner product? how about silly inner product $p \cdot q = \sum p_i q_i$

where $p = p_0 + p_1 t + \dots + p_4 t^4$ etc.

then want p s.t. $p \cdot 1 = 0$, $p \cdot t = 0$, $p \cdot t^2 = 0$, $p \cdot t^3 = 0$.

i.e. $p_0 = 0$ $\quad p_1 = 0$ $\quad p_2 = 0$ $\quad p_3 = 0$

so orthogonal complement has basis $\{t^4\}$.

Q6 $L:\mathbb{R}^2 \to \mathbb{R}^2 \quad L(x,y) = (x-2y,\ x+2y)$

a) $A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$

b) $P_{T\leftarrow S} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = P$

$$\begin{tikzcd} \mathbb{R}^2_T \arrow[r, "L"', "A"] \arrow[d, "P"'] & \mathbb{R}^2_T \arrow[d, "P"] \\ \mathbb{R}^2_S \arrow[r, "B"'] & \mathbb{R}^2_S \end{tikzcd}$$

$B = P \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} P^{-1}$

$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & -2 \\ 4 & 4 \end{bmatrix}$

c) $\text{rank}(A) = \text{rank}(\text{rref}(A)) = \text{rank}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$

$\text{rank}(A) + \text{nullity}(A) = 2 \Rightarrow \text{nullity} = 0.$

Q7

a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

b) $(\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta = 0$

$\lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta = 0$

$\lambda^2 - 2\lambda\cos\theta + 1 = 0$

$\lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta$

$\lambda = \cos\theta + i\sin\theta$: $\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}$ eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\lambda = \cos\theta - i\sin\theta$: $\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}$ eigenvector $\begin{bmatrix} 1 \\ i \end{bmatrix}$

Q8

a) $\det(A - \lambda I_3)$ is a cubic polynomial, always has at least one real root.

b) every rotation in $\mathbb{R}^3$ preserves a direction (axis).

c) $$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}$$

d) rotations in $\mathbb{R}^4$ don't need to fix a direction.