Linear Algebra Spring 10 Midterm 2

Name: Solutions

1	15	
2	15	
3	20	
4	10	
5	20	
6	10	
7	10	
	100	

(1) (15 points)

(a) Give an example of a spanning set in $\mathbb{R}^2$ which is not a basis. what can you say about the number of elements in such a set?

(b) Give an example of a linearly independent set in $\mathbb{R}^3$ which is not a basis. What can you say about the number of elements of such a set?

(c) If A is a 4×5 matrix, and the columns of A add up to the zero vector, what can you say about the nullity of A?

a) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ must be at least 3 elements.

b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ can have at most two elements.

c) column rank of a 4×5 matrix at most 4.

If columns sum to zero vector $\Rightarrow$ linearly dependent, so rank at most 3.

$$\underbrace{\text{rank}(A)}_{\leq 3} + \text{nullity}(A) = 4$$

so $\text{nullity}(A) \geq 1$

(2) (15 points)

Let S be the following set of vectors:

$$S = \{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \}$$

(a) Find a subset of S which is a basis for the span of S.
(b) Does the set S span $\mathbb{R}^3$?

a) solve $c_1 \underline{v}_1 + \cdots + c_4 \underline{v}_4 = \underline{0}$, row reduce $\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 2 & +1 & -1 \\ -1 & -1 & 0 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & 1 & 0 \\ -1 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

3rd column is free variable, so basis is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$

b) yes. (3 linearly independent vectors in $\mathbb{R}^3$ span).

(3) (20 points)

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & -1 \\ 1 & 1 & 3 & 2 & 2 \\ 3 & 1 & 7 & 8 & 0 \end{bmatrix}$$

(a) Find the reduced row echelon form for A.
(b) Find the rank and nullity of A.
(c) Find a basis for $\mathrm{Im}(A)$.
(d) Find a basis for $\mathrm{Ker}(A)$.

a) $\begin{bmatrix} 1 & 0 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 1 & 1 & -1 & 3 \end{bmatrix}$ $\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \end{array}$ $\begin{bmatrix} 1 & 0 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{array}{ccccc} & & r & s & t \end{array}$

b) $\mathrm{rank}(A) = 2$
$\mathrm{rank}(A) + \mathrm{nullity}(A) = 5$ so $\mathrm{nullity}(A) = 3$.

c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

d) solutions $x_5 = t$ $x_4 = s$ $x_3 = r$ $x_2 = -r + s - 3t$
$x_1 = -2r - 3s + t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r - 3s + t \\ -r + s - 3t \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

basis: $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(4) (10 points)

(a) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 1 & -3 \\ 3 & 1 & 2 \end{bmatrix}$$

Is the vector $(2, 4, 5)$ in the range of L? Justify.

solve $\underline{Ax = b}$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ -2 & 1 & -3 & 4 \\ 3 & 1 & 2 & 5 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 5 & -5 & 8 \\ 0 & -5 & 5 & -1 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 5 & -5 & 8 \\ 0 & 0 & 0 & 7 \end{array}\right]$ no solutions, no $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ <u>not</u> in range of L.

(5) (25 points)

Let $E = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$, $e_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], e_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, and let $S = \{u_1, u_2\}$, be the basis for $\mathbb{R}^2$ given by

$$u_1 = \begin{bmatrix}3\\1\end{bmatrix}, u_2 = \begin{bmatrix}1\\5\end{bmatrix}$$

(a) Verify that S is a basis.

(b) Find the coordinate vector of $v = \begin{bmatrix}-5\\3\end{bmatrix}$ with respect to the basis S.

(c) Find the transition matrix $P_{E\leftarrow S}$ from S to E, and use this to check your answer to (b).

(d) Find the transition matrix $P_{S\leftarrow E}$ from E to S.

a) $\begin{bmatrix}3 & 1\\1 & 5\end{bmatrix}$ row equivalent to $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ so $\{u_1, u_2\}$ is a basis.

b) $\underline{c_1 u_1 + c_2 u_2 = v}$ $\quad \left[\begin{array}{cc|c}3 & 1 & -5\\1 & 5 & 3\end{array}\right]$ $\quad \left[\begin{array}{cc|c}1 & 5 & 3\\0 & -14 & -14\end{array}\right]$

$\left[\begin{array}{cc|c}1 & 0 & -2\\0 & 1 & 1\end{array}\right]$ So $[v]_S = \begin{bmatrix}-2\\1\end{bmatrix}$

c) $P_{E\leftarrow S} = \begin{bmatrix}3 & 1\\1 & 5\end{bmatrix}$ $\quad \begin{bmatrix}3 & 1\\1 & 5\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}-5\\3\end{bmatrix}$ ✓.

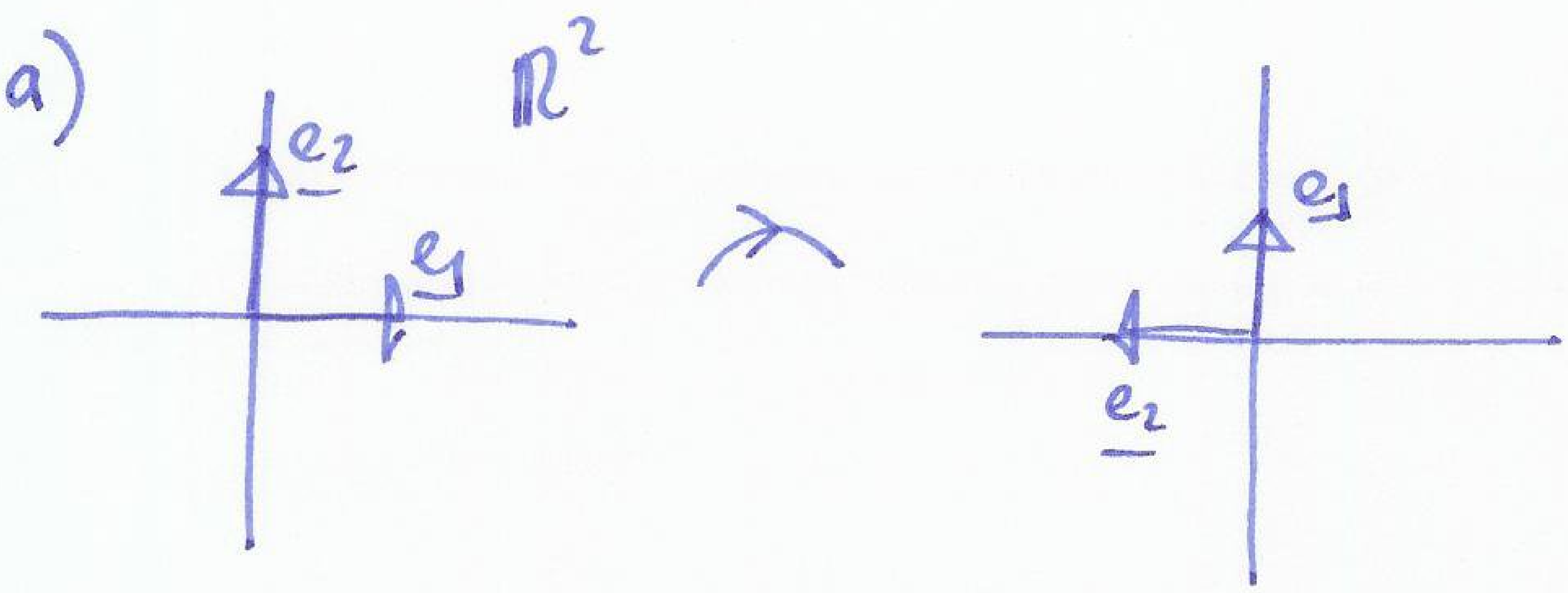
d) $P_{S\leftarrow E} = \begin{bmatrix}3 & 1\\1 & 5\end{bmatrix}^{-1} = \frac{1}{14}\begin{bmatrix}5 & -1\\-1 & 3\end{bmatrix}$

(6) (10 points)

(a) Write down the matrix for a linear transformation of $\mathbb{R}^2$ that reflects in the y-axis and halves lengths. (Hint: Draw a picture.)

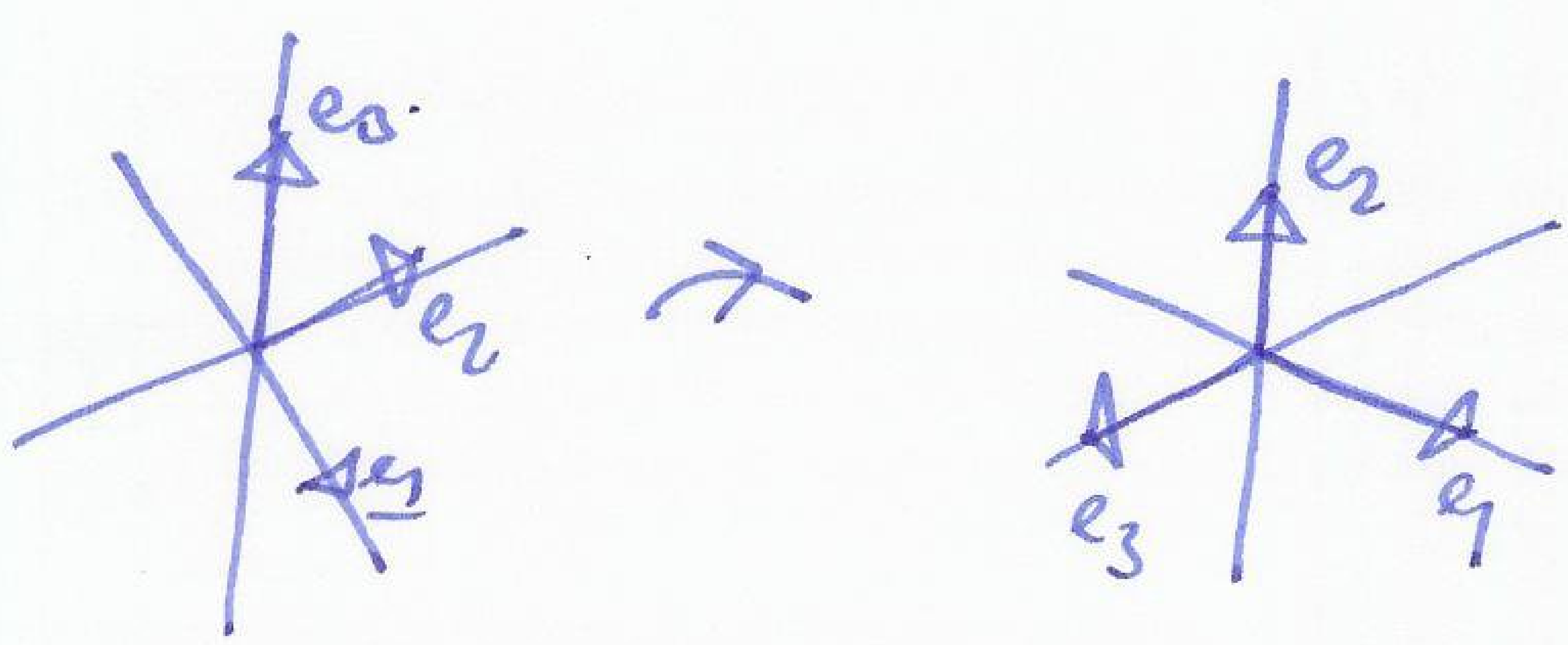
(b) Write the matrix for a linear transformation of $\mathbb{R}^3$ that makes a $\pi/2$ rotation in the yz-plane.

a)



$$\begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}$$

b)



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

(7) (10 points)

Let v be a vector in $\mathbb{R}^n$. Show that the set of all vectors perpendicular to v is a vector space.

check closed under addition:

$\underline{v}.\underline{x} = 0$ then $\underline{v}.(\underline{x}+\underline{y}) = \underline{v}.\underline{x} + \underline{v}\underline{y} = 0$

$\underline{v}.\underline{y} = 0$

check closed under scalar multiplication:

$\underline{v}.\underline{x} = 0$ then $\underline{v}.(\lambda\underline{x}) = \lambda\, \underline{v}.\underline{x} = \lambda 0 = 0.$