Linear Algebra Sample Final

Q1 a) Yes Always (uses Jordan normal form which we didn't do)

b) Yes Always $\underset{\leq 3}{\text{rank}(A)} + \text{nullity}(A) = \underset{n}{\#\text{cols}}$. so $\text{nullity}(A) \geq 2$

c) Sometimes. true if $A = 0$ matrix, not true in general, e.g.

$$\underset{\det 0}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} + \underset{\det 1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{\det 3}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}.$$

d) Always as $\det(AB) = \det(A)\det(B)$.

e) Never, two identical cols $\Rightarrow \det = 0$, so not invertible.

Q2 a) $\det(AB) = \det(A)\det(B)$ so $\det(AA^{-1}) = \det(A)\det(A^{-1})$

$\qquad \det(I_n) = 1$.

so $\det(A^{-1}) = \dfrac{1}{\det(A)}$

b) $\det(A) \neq 0$ $\det(B) \neq 0 \Rightarrow \det(A)\det(B) \neq 0 \Rightarrow \det(AB) \neq 0$

$\therefore AB$ invertible.

c) suppose B, C are inverses for A, then $AB = I = BA$

$AC = I = CA$

so $AB = AC \Rightarrow BAB = BAC \Rightarrow IB = IC \Rightarrow B = C$.

Q3 a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

b) $0x + 0y + 0z = 1$

$0x + 0y + 0z = 0$

c) impossible.

②

d) $0x + 0y + 0z = 0$

$0x + 0y + 0z = 0.$

Q4 $\left[\begin{array}{cccc|c} 2 & -1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 1 \\ 0 & -1 & 2 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3/4 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1 \\ 0 & 0 & 1 & 1/4 & 1/2 \end{array}\right] \\ \begin{array}{cccc} & & & t \end{array} \end{array}$

$x_4 = t$

$x_3 = 1/2 - 1/4\,t$ $\qquad x = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3/4 \\ -1/2 \\ -1/4 \\ 1 \end{bmatrix}$

$x_2 = 1 - 1/2\,t$

$x_1 = 1/2 - 3/4\,t$

Q5 a) $A+B = \begin{bmatrix} 4 & 1 & 3 \\ -1 & 0 & 8 \\ 1 & 3 & 1 \end{bmatrix}$ $\qquad AB = \begin{bmatrix} 18 & 4 & 3 \\ 26 & 4 & -3 \\ -5 & 3 & 6 \end{bmatrix}$

$B^T = \begin{bmatrix} 2 & -1 & 5 \\ 0 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ $\qquad \cancel{\det(A) = 1 \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} - 8 \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = 1 - 88 = -87}$

$\det(A) = 2 \begin{vmatrix} -1 & 5 \\ 2 & 1 \end{vmatrix} - 4 \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} = 2(-11) - 4(8) = -54$

$\det(A^T) = -54. \quad \det(3A) = 3^3 \det(A) = -27 \cdot 54 = -1458.$

b) $\det C = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$

Q6 $\left[\begin{array}{cc|c} 4+\epsilon & 2 & 1 \\ 1.1 & 2 & 1 \end{array}\right]$ where $-0.4 \le \epsilon \le 0.4$ (10% error).

$$\det\begin{bmatrix} 4+\epsilon & 2 \\ 1.1 & 2 \end{bmatrix} = 8+2\epsilon - 2.2 = 5.8+2\epsilon.$$

$$x = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}}{5.8+2\epsilon} = 0 \qquad y = \frac{\begin{vmatrix} 4+\epsilon & 1 \\ 1.1 & 1 \end{vmatrix}}{5.8+2\epsilon} = \frac{2.9+\epsilon}{5.8+2\epsilon}.$$

so $\dfrac{2.9-0.4}{5.8+2\cdot 0.4} \le y \le \dfrac{2.9+0.4}{5.8-2\cdot 0.4}$ error approx 32%.

$\approx 0.38 \qquad\qquad \approx 0.66$

$\left[\begin{array}{cc|c} 4+\epsilon & 2 & 1 \\ 2 & 1.1 & 1 \end{array}\right]$ $-0.4 \le \epsilon \le 0.4$ $\det\begin{bmatrix} 4+\epsilon & 2 \\ 2 & 1.1 \end{bmatrix} = 1.1(4+\epsilon) - 4$

$= 0.1 + 1.1\epsilon.$

$$x = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 1.1 \end{vmatrix}}{0.1+1.1\epsilon} = \frac{-0.9}{0.1+1.1\epsilon} \leftarrow \text{arbitrarily large error!}$$

as denominator may be arb. close to zero!

Q7 a) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & -5 & 1 & 2 \\ 3 & 2 & 4 & 3 \end{array}\right] \overset{\text{rref}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ yes.

b) $A = \begin{bmatrix} 2 & 3 \\ -2 & 3 \\ 1 & 1 \end{bmatrix}$

Q8 a) at least 7 vectors.

b) $\text{rank}(A) + \text{nullity}(A) = \#\text{cols} = n$

$\text{rank}(A) < n \Rightarrow \text{nullity}(A) > 0 \Rightarrow \infty$-many solutions (at least 1d family).

c) orthogonal $\Rightarrow$ linearly independent $\Rightarrow \text{rank}(A) = n$.

d) T orthogonal (orthonormal if you normalize) S not nec. orthogonal.

e) $\text{rank}(A) + \text{nullity}(A) = 3$ sum of cols $= \underline{0}$ $\Rightarrow \text{rank}(A) \le 2$

$\Rightarrow \text{nullity}(A) \ge 1$.

Q9
$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \underset{\text{rref}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \right\} \text{ basis}$$

for span. dim span = 3.

Q10 $\{ \overset{\underline{w_1}}{1}, \overset{\underline{w_2}}{t}, \overset{\underline{w_3}}{t^2} \}$.

$\underline{v_1} = 1$.

$\underline{v_2} = t - \lambda 1$ where $\underline{v_1}\cdot\underline{v_2} = 0 = \underline{v_1}\cdot\underline{w_2} - \lambda\, \underline{v_1}\cdot\underline{v_1}$

so $\lambda = \dfrac{\underline{v_1}\cdot\underline{w_2}}{\underline{v_1}\cdot\underline{v_1}}$ $\quad \underline{v_1}\cdot\underline{v_1} = \int_{-1}^{1} 1\, dt = [t]_{-1}^{1} = 2$.

$\underline{v_1}\cdot\underline{w_2} = \int_{-1}^{+1} t\, dt = \left[\frac{1}{2}t^2\right]_{-1}^{1} = 0$

so $\underline{v_2} = t$

$\underline{v_3} = t^2 - \lambda_1 1 - \lambda_2 t$ where $\lambda_1 = \dfrac{\underline{v_1}\cdot\underline{w_3}}{\underline{v_1}\cdot\underline{v_1}}$ $\lambda_2 = \dfrac{\underline{v_2}\cdot\underline{w_3}}{\underline{v_2}\cdot\underline{v_2}}$

$\underline{v_1}\cdot\underline{w_3} = \int_{-1}^{1} 1.t^2\,dt = \left[\frac{1}{3}t^3\right]_{-1}^{1} = \frac{2}{3}$ so $\lambda_1 = \frac{2/3}{2} = 1/3$.

$\underline{v_2}\cdot\underline{w_3} = \int_{-1}^{1} t.t^2\,dt = \left[\frac{1}{4}t^4\right]_{-1}^{1} = 0$ so $\lambda_2 = 0$.

so $\underline{v_3} = t^2 - \frac{1}{3}$

Q11 a) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$tr(A) = a_{11} + a_{22}$ $tr(A+B) = a_{11} + b_{11} + a_{22} + b_{22}$ ✓.

$tr(B) = b_{11} + b_{22}$

$tr(\lambda A) = \lambda a_{11} + \lambda a_{22} = \lambda(tr(A))$ ✓.

b) $S = \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$. $tr(\underline{v_1}) = 1$

$tr(\underline{v_2}) = 0$

$tr(\underline{v_3}) = 1$

so $A = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ ($A = tr$ wrt S).

c) $T = \{\underline{w_1}, \underline{w_2}, \underline{w_3}\}$ where $\underline{w_1} = -\underline{v_1} + \underline{v_2}$

$\underline{w_2} = \underline{v_1} + \underline{v_2} + \underline{v_3}$

$\underline{w_3} = \underline{v_1} + \underline{v_3}$.

so $P_{S \leftarrow T} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

tr wrt $T = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}$.

Q12

a) $E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ then $P_{E \leftarrow S} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$

so $[\underline{v}]_S = \Big\{ P_{E\leftarrow S}^{-1} \begin{bmatrix} -1 \\ 8 \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} 2 & -3 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 8 \end{bmatrix} = \frac{-1}{13} \begin{bmatrix} -26 \\ 13 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

b) $P_{E \leftarrow T} = P_{E \leftarrow S} P_{S \leftarrow T} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 7 & -4 \end{bmatrix}$

so $T = \left\{ \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ -4 \end{bmatrix} \right\}.$

Q13 a) eigenvalues: $(1-\lambda)(-\lambda) + 1 = 0$

$$\lambda^2 - \lambda + 1 = 0$$

$\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$ $\quad \lambda_1 = \frac{1+\sqrt{3}i}{2} \quad \lambda_2 = \frac{1-\sqrt{3}i}{2}.$

eigenvectors: $\underline{v_1}$: $\left[\begin{array}{cc|c} \frac{1}{2} - \frac{\sqrt{3}i}{2} & -1 & 0 \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}i}{2} & 0 \end{array}\right]$ $\quad \underline{v_1} = \cancel{\begin{bmatrix} \frac{1}{2} - \frac{\sqrt{3}i}{2} \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ \frac{1}{2} - \frac{\sqrt{3}i}{2} \end{bmatrix}.$

$\underline{v_2}$: $\left[\begin{array}{cc|c} \frac{1}{2} + \frac{\sqrt{3}i}{2} & -1 & 0 \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}i}{2} & 0 \end{array}\right]$ $\quad \underline{v_2} = \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{3}i}{2} \end{bmatrix}.$

b) we $\lambda_1^6 = \lambda_2^6 = 1$ so $A^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as two distinct +1 eigenvectors.

Q4 a) $L_E = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

$S = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ $\qquad P_{E \leftarrow S} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} = P.$

$L_S = P_{S \leftarrow E} L_E P_{E \leftarrow S} = P^{-1} L_E P.$

$$L_S = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$