observation ① choice of antiderivative doesn't matter.

spose $f(x)$ has antiderivatives $F(x)$ and $F(x)+C$

then $\int_a^b f(x)dx = F(b)-F(a)$

$$\int_a^b f(x)dx = F(b)+C-(F(a)+C) = F(b)-F(c).$$

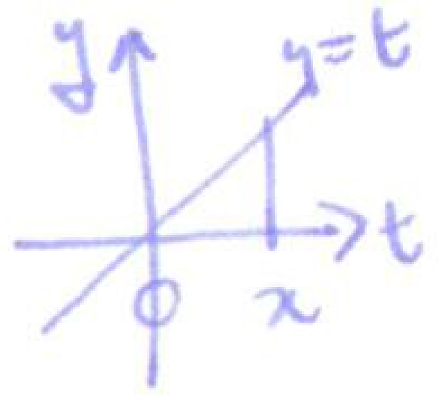
② $\int_a^b f(x)dx$ is a function of a and b; x is a dummy variable: $\int_a^b f(x)dx = \int_a^b f(t)dt$.

§5.4 Fundamental theorem of calculus ②

Thm: FTC ② Let $f(x)$ be cts on $[a,b]$ then $A(x) = \int_a^x f(t)\,dt$ is an anti-derivative for $f(x)$, i.e. $A'(x) = \frac{dA}{dx} = f(x)$

i.e. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and furthermore $A(a)=0$.

Examples



$$\int_0^x t\,dt = \left[\tfrac{1}{2}t^2\right]_0^x = \tfrac{1}{2}x^2 - 0 = \tfrac{1}{2}x^2.$$

$\int_0^x e^{-t^2}dt$ ← a function with derivative e^{-t^2}

observation

$$f(x) \xrightarrow{\text{integrate}} F(x) \xrightarrow{\text{differentiate}} f(x)$$

$$\int_a^x f(t)\,dt \qquad \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

$$f(x) \xrightarrow{\text{differentiate}} f'(x) \xrightarrow{\text{integrate}} \int_a^x f'(t)\,dt = f(x)-f(a)$$

↑ only equal to $f(x)$ up to a constant.

Final
Thu 16
1:30 - 3:30 15-102

<u>warning</u> what about $\int_0^{x^2} \sin t \, dt$?

set $A(x) = \int_0^x \sin t \, dt$ then $G(x) = \int_0^{x^2} \sin t \, dt$

is $G(x) = A(x^2)$.

so $\frac{d}{dt} \int_0^{x^2} \sin t \, dt = \frac{d}{dt}\left(A(x^2)\right)$ use chain rule $= A'(x^2) \cdot 2x$.

$= 2x \sin x$.

§5.5 Net change (applications of integrals)

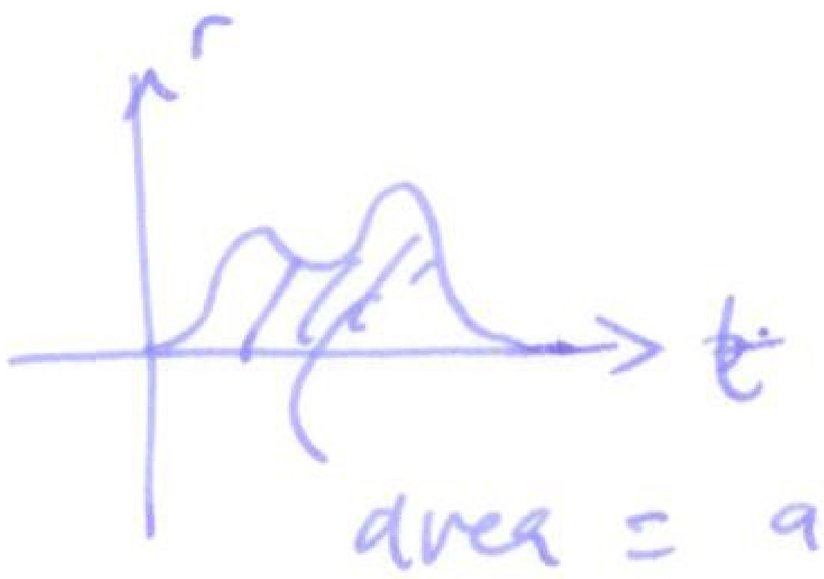
<u>Example</u> you pour water into a bucket at rate $r(t)$. How much water is in the bucket?

constant rate: amount = rate × time.

area =amount of water

varying rate $r(t)$

amount $= \int_0^t r(x) dx$



area = amount of water.

how much did the amount of water change between times t_1 and t_2?

by

net change in amount of water $= \int_{t_1}^{t_2} r(x) dx = \int_{t_1}^{t_2} r(t) dt$

• traffic/queueing

amount in $i(t)$

amount out $o(t)$

rate of change ~~at~~ $i(t) - o(t)$

net change $\int_{t_1}^{t_2} i(t) - o(t) dt$.

• distance as integral of velocity

distance travelled $\int_{t_1}^{t_2} v(t) dt$.

§5.6 Substitution / change of variable

"reverse chain rule for integration"

recall: chain rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\, g'(x)$

Example $\frac{d}{dx}\left(\sin(x^2)\right) = \cos(x^2) \cdot 2x.$

so $\int 2x\cos(x^2)\,dx = \sin(x^2) + C$

substitution / change of variable. $\int f(u)\,du \qquad u(x)$

$$\boxed{\int_a^b f(u)\,du = \int_{u^{-1}(a)}^{u^{-1}(b)} f(u(x))\frac{du}{dx}\,dx}$$

remember: three things to change: function, limits, differential.

mnemonic: $du = \frac{du}{dx}\,dx$

why does this work?: $\int f(u)\,du \qquad u(x)$

set $F(u) = \int f(u)\,du$ so $F'(u) = f(u)$

$$\int f(u)\,du = F(u) \underset{\text{wrt } x}{\overset{\text{differentiate}}{\rightsquigarrow}} \frac{d}{dx}\left(F(u(x))\right) = F'(u(x)) \cdot u'(x)$$

$$= f(u(x))\,u'(x).$$

$$\underset{\text{wrt } x}{\overset{\text{integrate.}}{\rightsquigarrow}} \int f(u(x))\,u'(x)\,dx = \int f(u(x))\frac{du}{dx}\,dx. \square$$