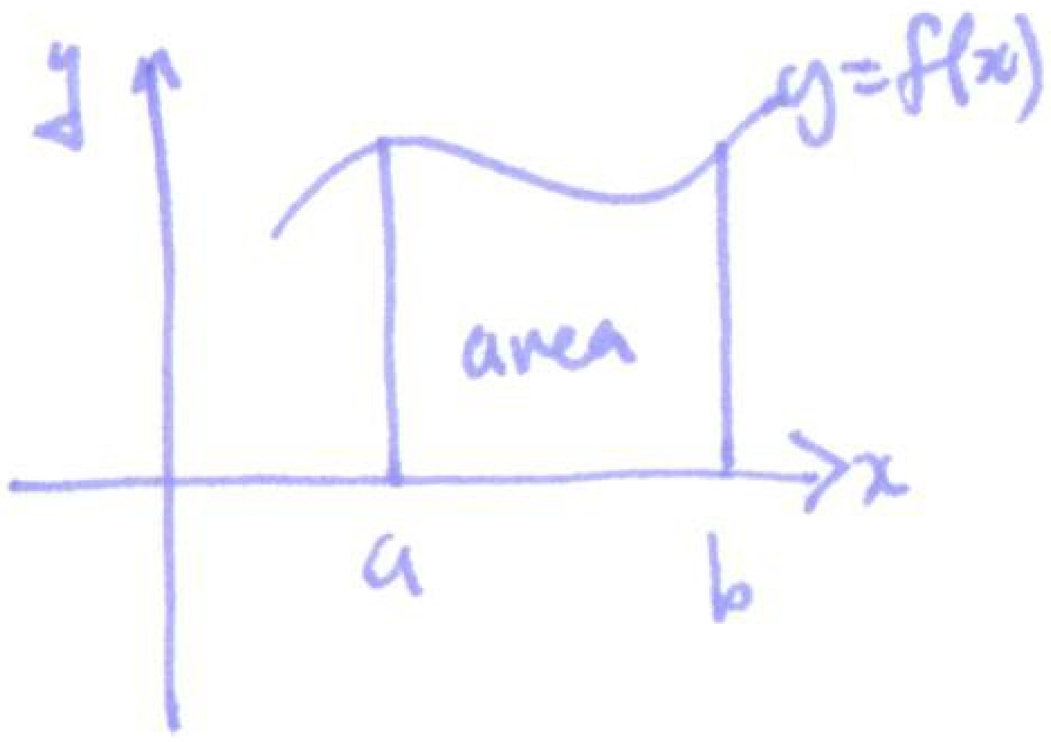
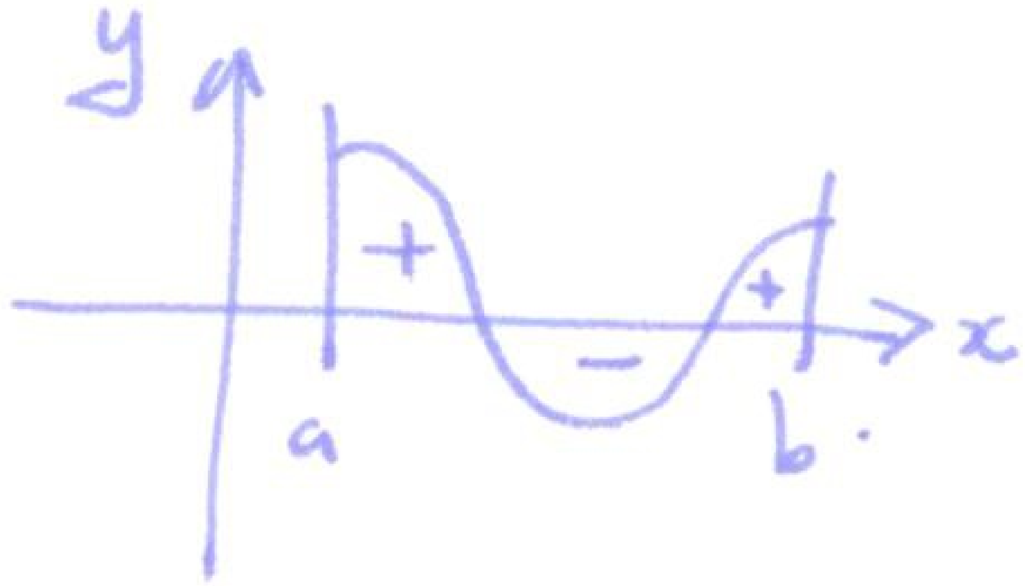
## §5.2 Definite integral

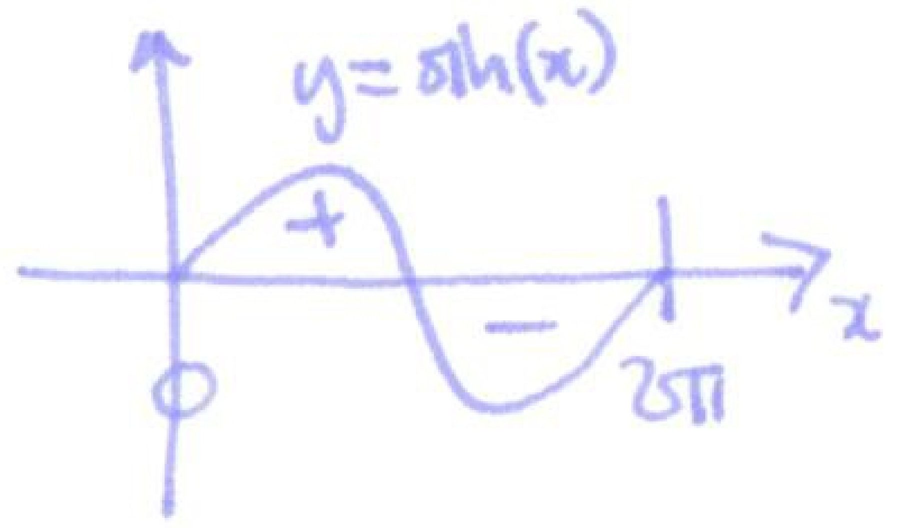



$$\int_a^b f(x)\,dx = \text{area under curve } y=f(x) \text{ between } x=a \text{ and } x=b.$$

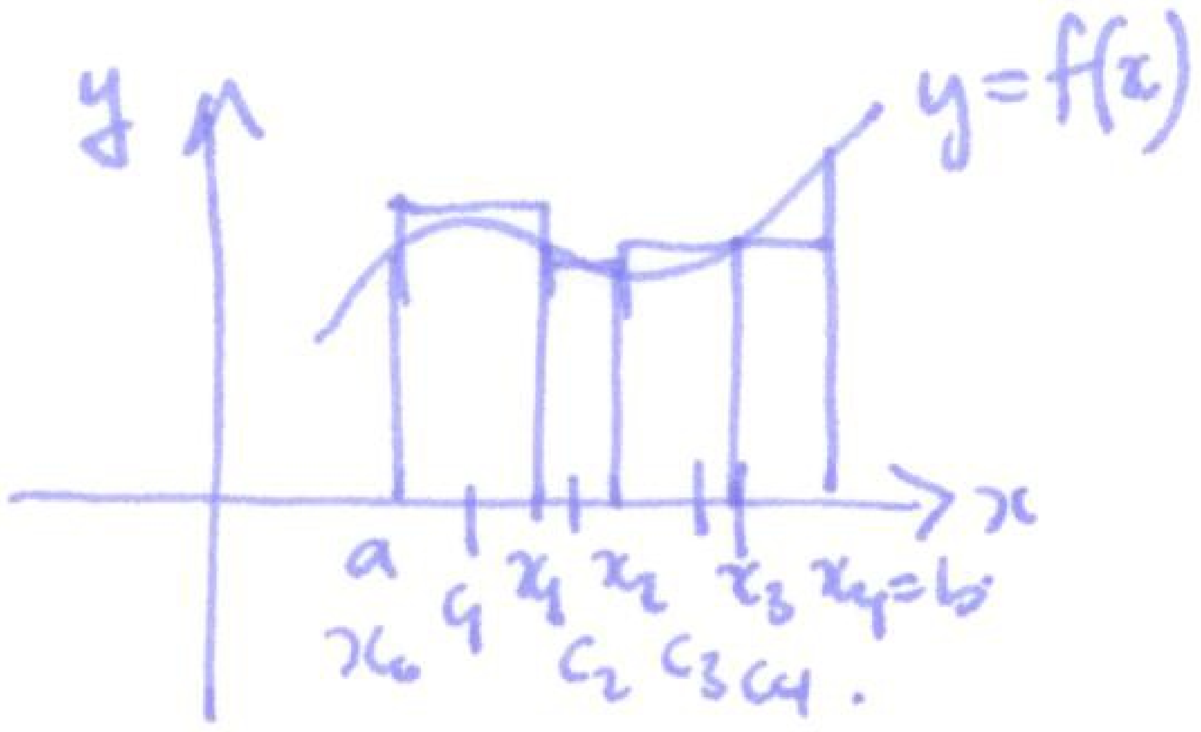
note: signed area.




so




$$\int_0^{2\pi} \sin(x)\,dx = 0.$$




Formal definition: Riemann sum

$R(f, P, c)$

$P$: partion of $[a,b]$ sequence of points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

$c$: choice of points $c_i \in [x_{i-1}, x_i]$

$$R(f,P,c) = \sum_{i=1}^{n} f(c_i)\Delta x_i \qquad \text{where } \Delta x_i = x_i - x_{i-1}.$$

$$\|P\| = \max \Delta_i$$

Def$^{\underline{n}}$
$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} R(f,P,c) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

when this limit exists we say $f$ is integrable over $[a,b]$.

## useful properties

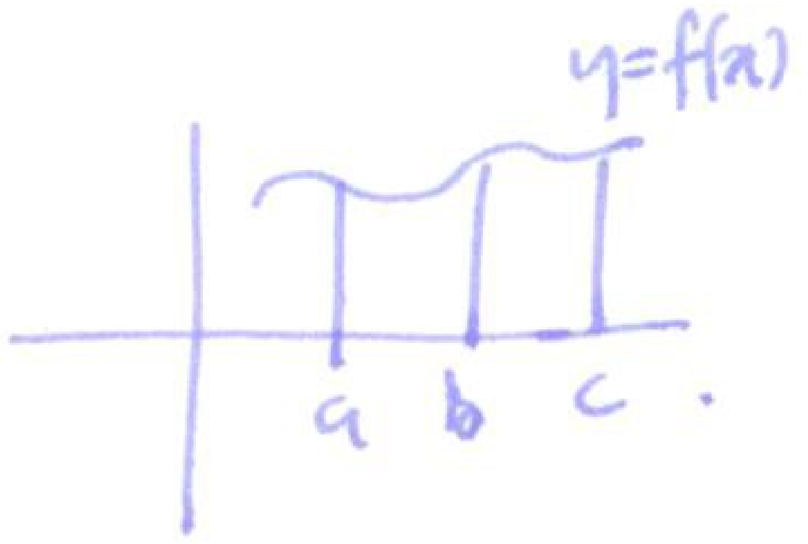
$$\int_a^b c\,dx = c(b-a)$$

$$\int_a^b f(x)\,dx + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

0-length integral: $\int_a^a f(x)\,dx = 0$




adjacent intervals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

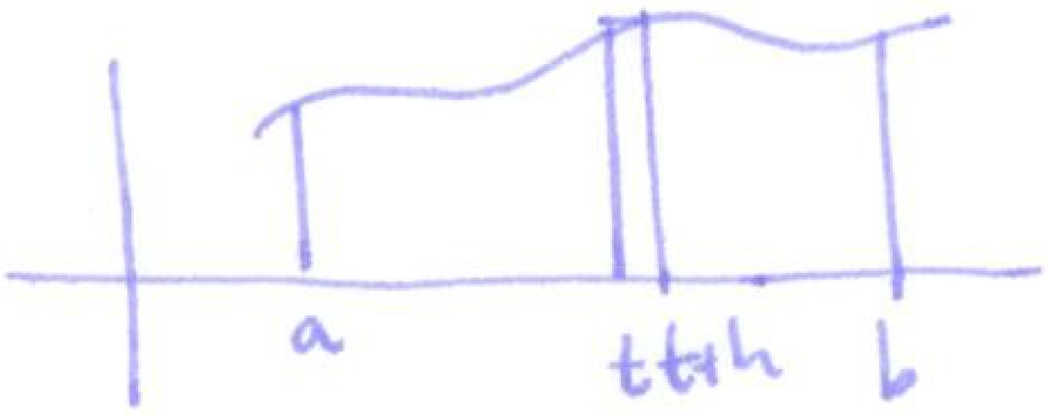
comparisons: if $f(x) \leq g(x)$ on $[a,b]$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$

## §5.3 Fundamental theorem of calculus

**Thm** FTC① spose $f(x)$ is cts on $[a,b]$ and $F(x)$ is an antiderivative for $f(x)$

then $\int_a^b f(x)\,dx = F(b) - F(a)$

intuition:



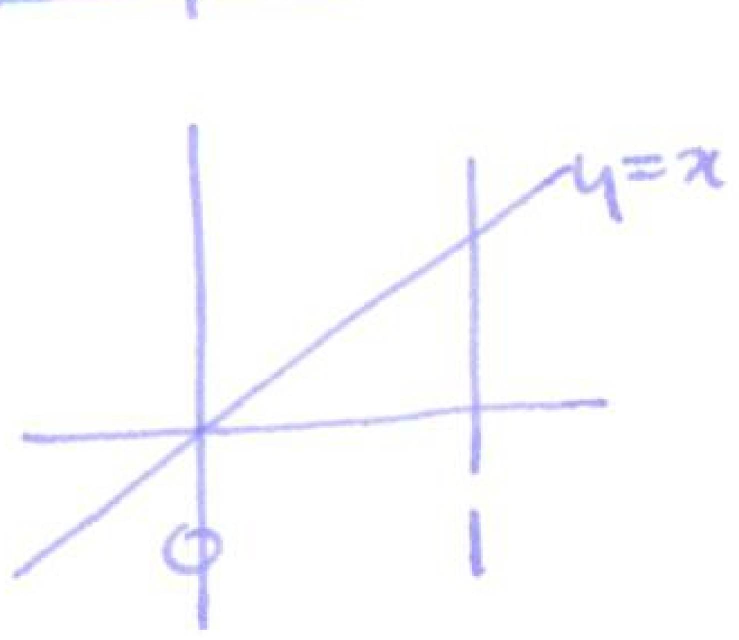

consider $\int_a^t f(x)\,dx = F(t) - F(a)$

Q what's the rate of change wrt t? $\frac{d}{dt}\left(F(t) - F(a)\right) = F'(t)$

varying $t \leftrightarrow$ adding on a thin rectangle.
of area $\approx h \times f(t)$ $\quad \lim_{h\to 0} \frac{F(t+h) - F(t)}{h} \approx \lim_{h\to 0} \frac{F(t) + hf(t) - F(t)}{h} = f(t)$

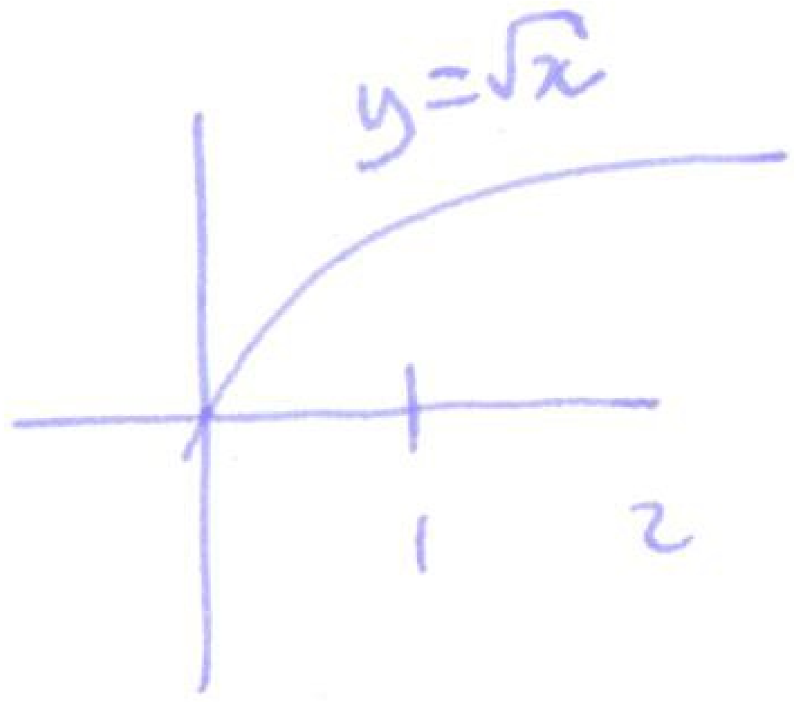
so $f(t)$ is the derivative of $F(t)$ i.e. $F(t)$ is an antiderivative of $f(t)$.

Examples

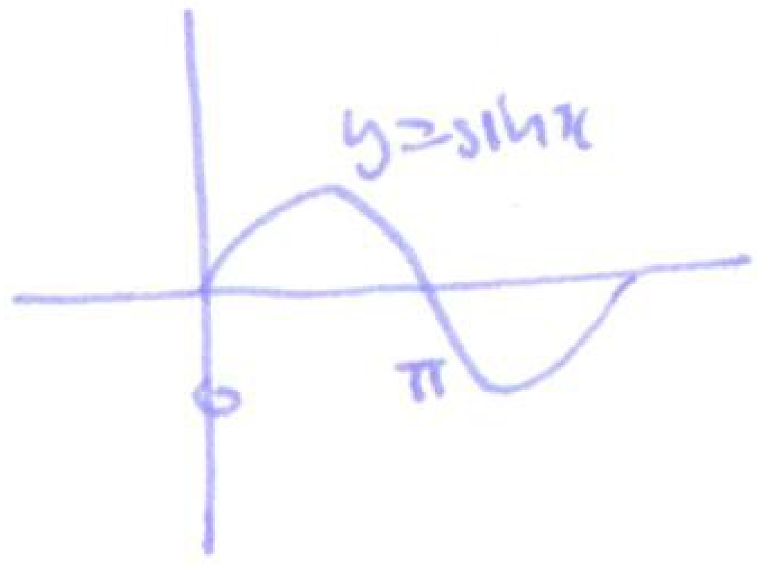



$$\int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2} - 0 = \tfrac{1}{2}.$$

$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}.$$

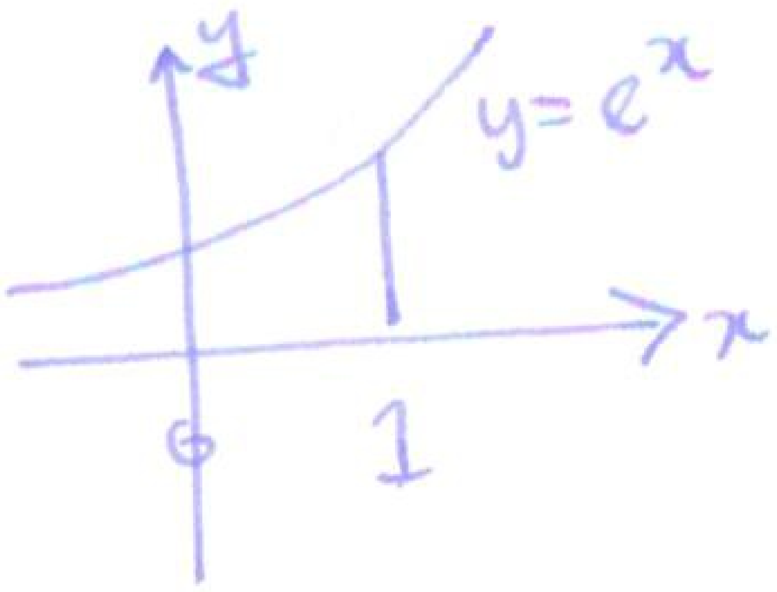



$$\int_1^2 \sqrt{x}\,dx = \left[\frac{2x^{3/2}}{3}\right]_1^2 = \frac{2}{3}\left(2^{3/2} - 1\right).$$

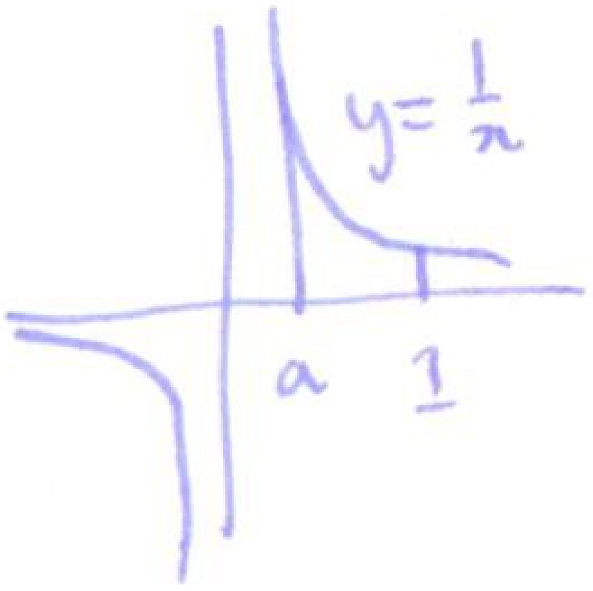



$$\int_0^{\pi} \sin(x)\,dx = \left[-\cos x\right]_0^{\pi} = -\cos(\pi) + \cos(0)$$
$$= -(-1) + 1 = 2.$$

$$\int_0^{2\pi} \sin(x)\,dx = \left[-\cos(x)\right]_0^{2\pi} = -\cos(2\pi) + \cos(0).$$
$$= -1 + 1 = 0.$$




$$\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1$$




$$\int_a^1 \frac{1}{x}\,dx = \left[\ln|x|\right]_a^1 = \ln 1 - \ln(a)$$
$$= -\ln(a)$$
$$\to \infty \text{ as } a \to 0.$$