Alternate view

we can think of finding the antiderivative as solving a differential equation $\frac{dy}{dx} = f(x)$.

in general there is a family of solutions $F(x)+c$.

but if we know a particular value of the solution we want (sometimes called an initial condition) we get a particular solution.

Example an object falls freely under gravity, so

acceleration $a(t) = -g$ (constant).

velocity $v(t)$ then $v'(t) = a(t)$ so $v'(t) = -g$

so $v(t) = \int -g \, dt = -gt + c$

if the velocity at time zero is v_0 then want

$$v(0) = -g.0 + c = v_0 \quad \text{i.e. } c = v_0.$$

so $v(t) = v_0 - gt$

now find position: $x(t)$ $x'(t) = v(t)$

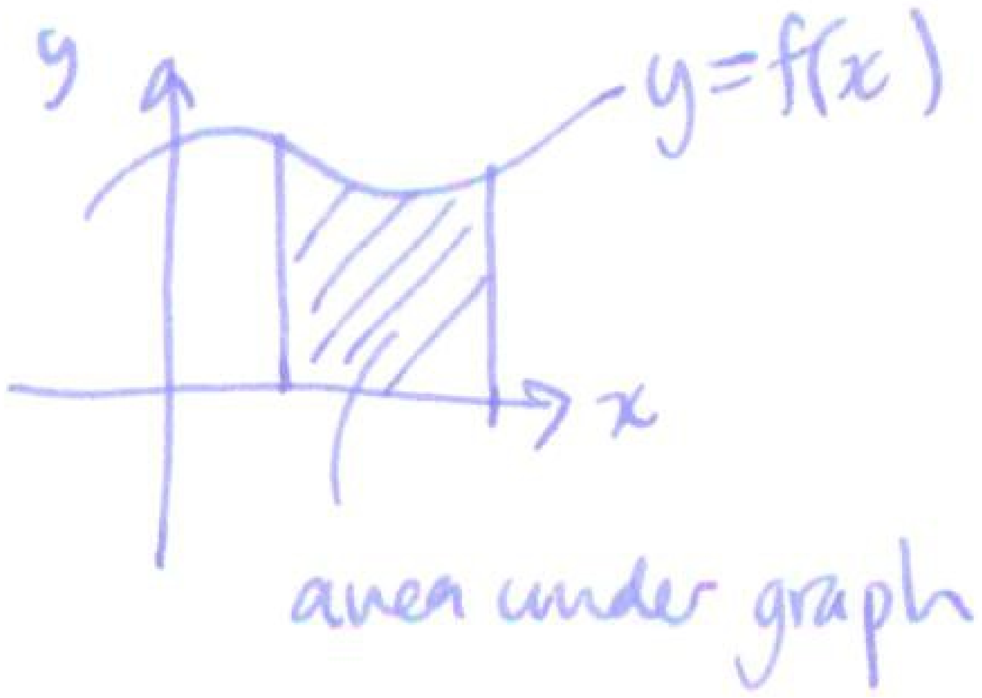
so $x(t) = \int v_0 - gt \, dt = v_0 t - \frac{1}{2}gt^2 + c$

if the ~~velocity~~ position at time zero is x_0 then

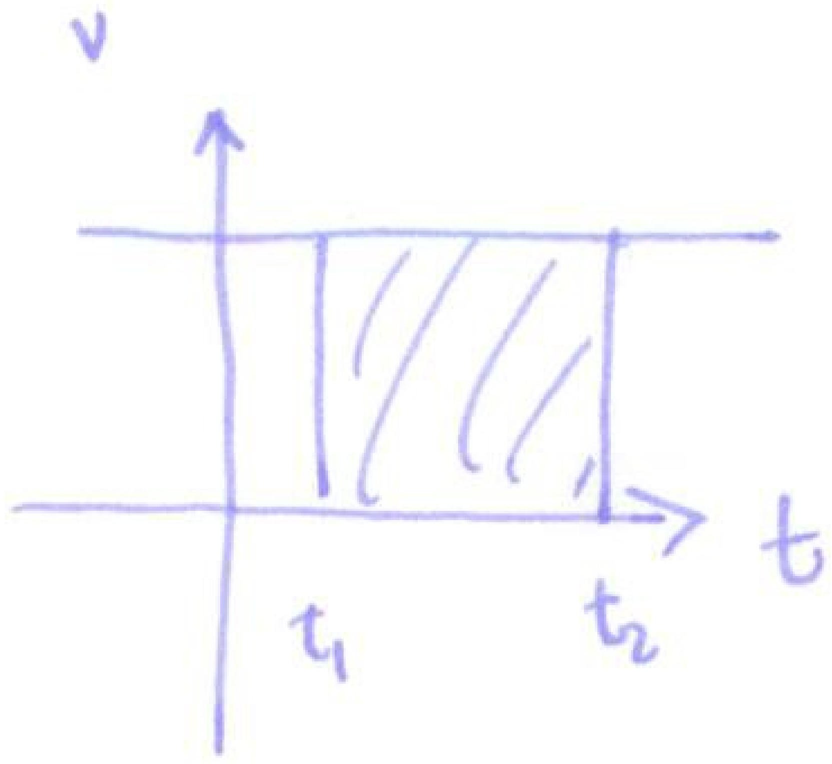
$$x(0) = v_0.0 - \frac{1}{2}g.0^2 + c = c = x_0$$

so $x(t) = x_0 + v_0 t - \frac{1}{2}gt^2$.

§5.1 Approximating areas



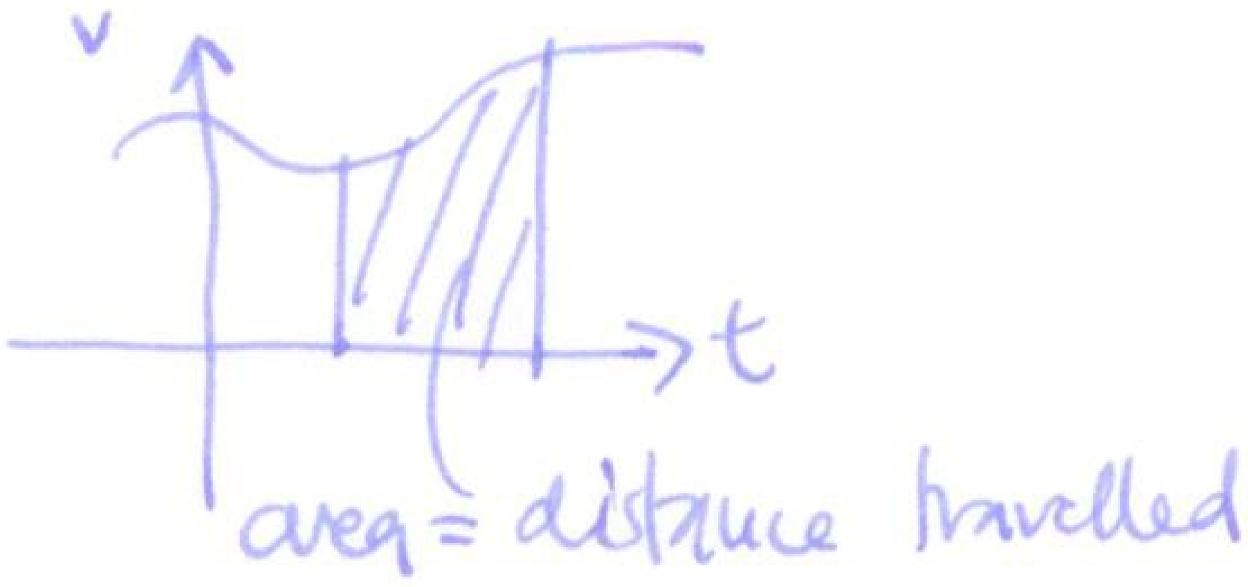
eg:



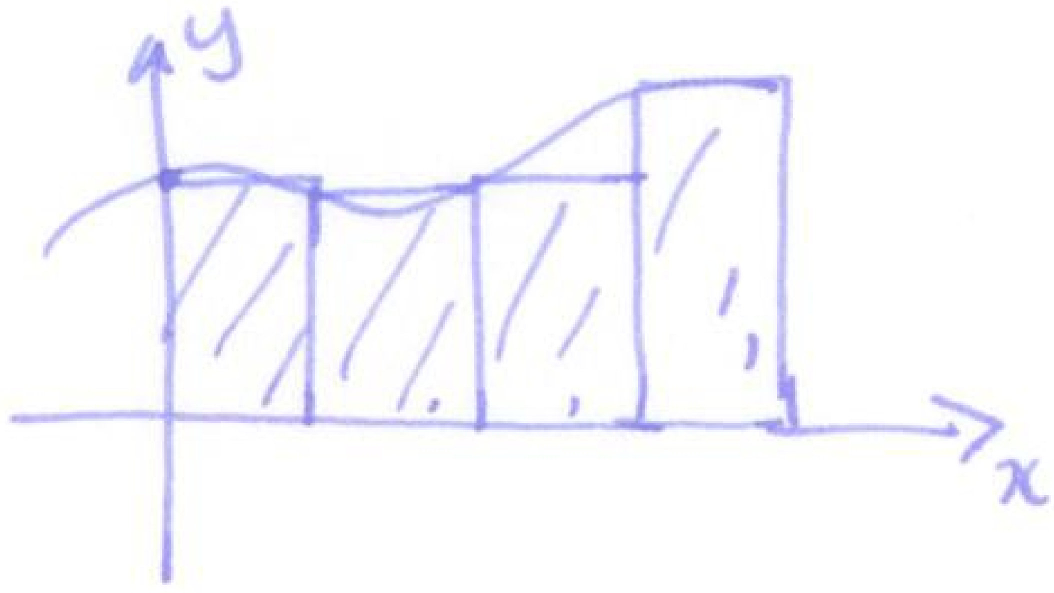
constant velocity

distance travelled = velocity × time

= area under velocity graph

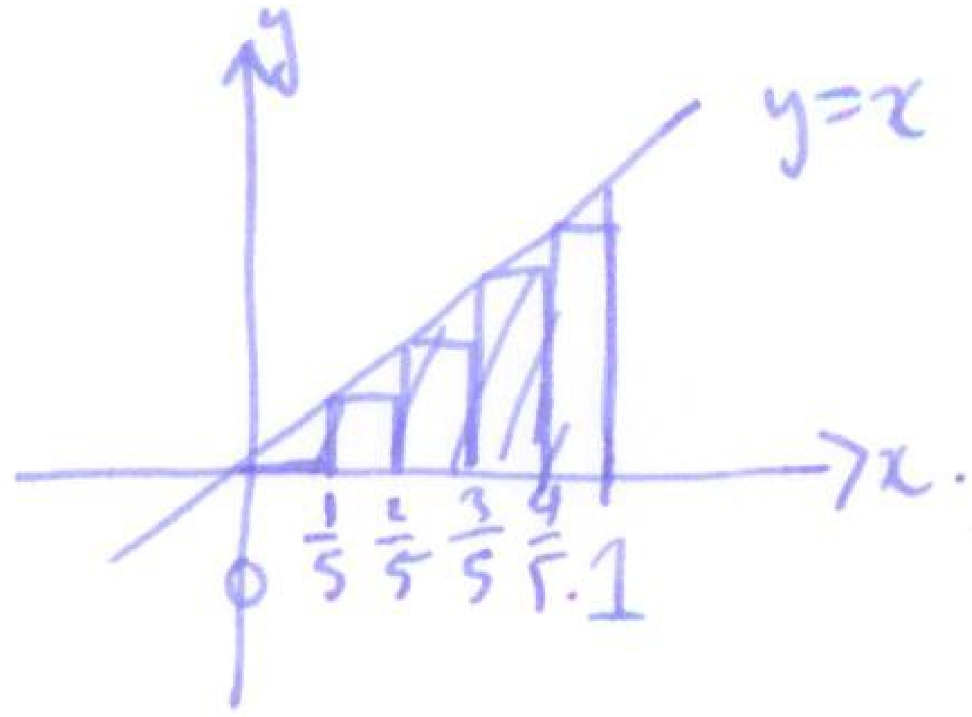


finding the area:



approximate area by rectangles.

Example



find area under $y=x$ between 0 and 1
(answer $= \frac{1}{2}$)

five rectangles: $\cancel{f(0)}$ $\sum$ rectangles. $= \sum f(x)$.width.

$$= f(0)\cdot\frac{1}{5} + f\left(\frac{1}{5}\right)\frac{1}{5} + f\left(\frac{2}{5}\right)\frac{1}{5} + f\left(\frac{3}{5}\right)\frac{1}{5} + f\left(\frac{4}{5}\right)\frac{1}{5}$$

$$= \frac{1}{5}\left(0 + \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right) = \frac{1}{25}\cdot(10) = \frac{10}{25}.$$

n rectangles.

$$f(0)\frac{1}{n} + f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \cdots + f\left(\frac{n-1}{n}\right)\frac{1}{n}.$$

$$= \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$$

$$= \frac{1}{n} \sum_{k=1}^{n-1} \frac{k}{n} = \frac{1}{n^2} \sum_{k=0}^{n-1} k$$

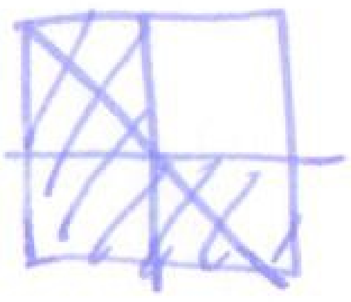
note $S_n = 1 + 2 + 3 + \cdots + N = \frac{1}{2}N(N+1)$

Proof ① induction. spose true for k: $S_k = 1+2+3+\cdots+k$

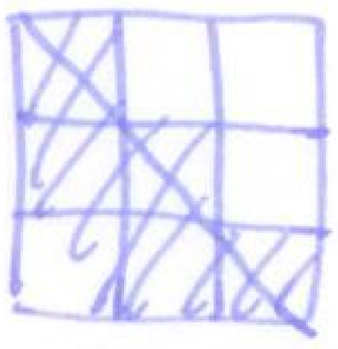
what about S_{k+1}? $S_{k+1} = \underbrace{1+2+3+\cdots+k}_{\frac{1}{2}k(k+1)} + k+1$

$$S_{k+1} = \tfrac{1}{2}k(k+1) + (k+1) = (k+1)\left(\tfrac{1}{2}k+1\right) = \tfrac{1}{2}(k+1)(k+2) \checkmark.$$

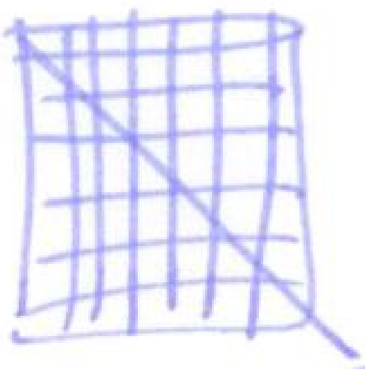
②



$\frac{1}{2}(2)^2 + \frac{1}{2}(2)$.



$\frac{1}{2}(3)^2 + \frac{1}{2}(3)$.

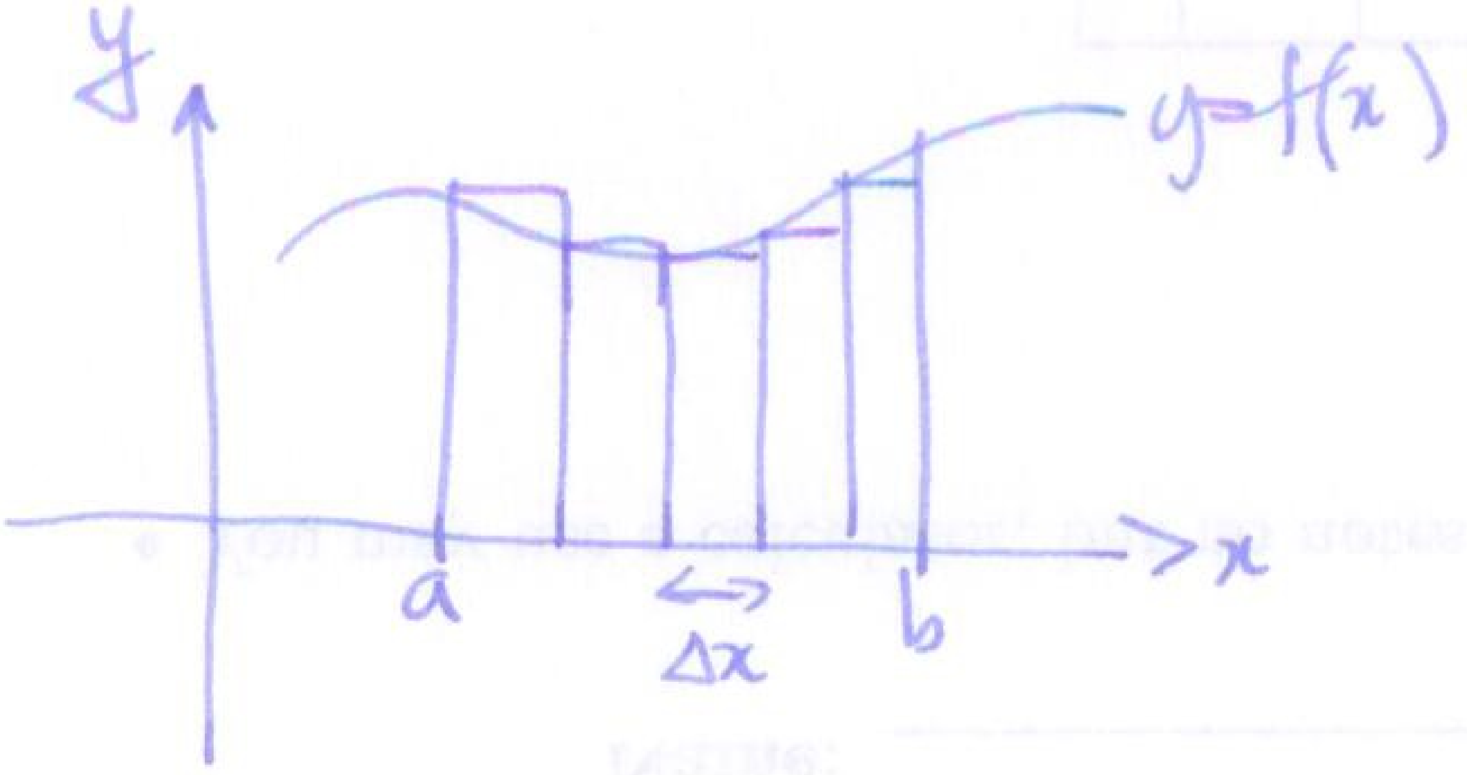


$\frac{1}{2}(N^2) + \frac{1}{2}N = \frac{1}{2}N(N+1)$

so approx area $= \frac{1}{n^2} \frac{1}{2}(n-1)n = \frac{1}{2}\frac{n-1}{n} = \frac{1}{2}\left(1 - 1/n\right)$.

so as $n \to \infty$ approx area $\to \frac{1}{2}$ = actual area.

Notation



spose there are N rectangles (i.e. $\Delta x = \frac{b-a}{N}$).

then (~~right~~ left endpoint) rectangle approx $L_N = \Delta x \sum_{j=0}^{N-1} f(a + j\Delta x)$

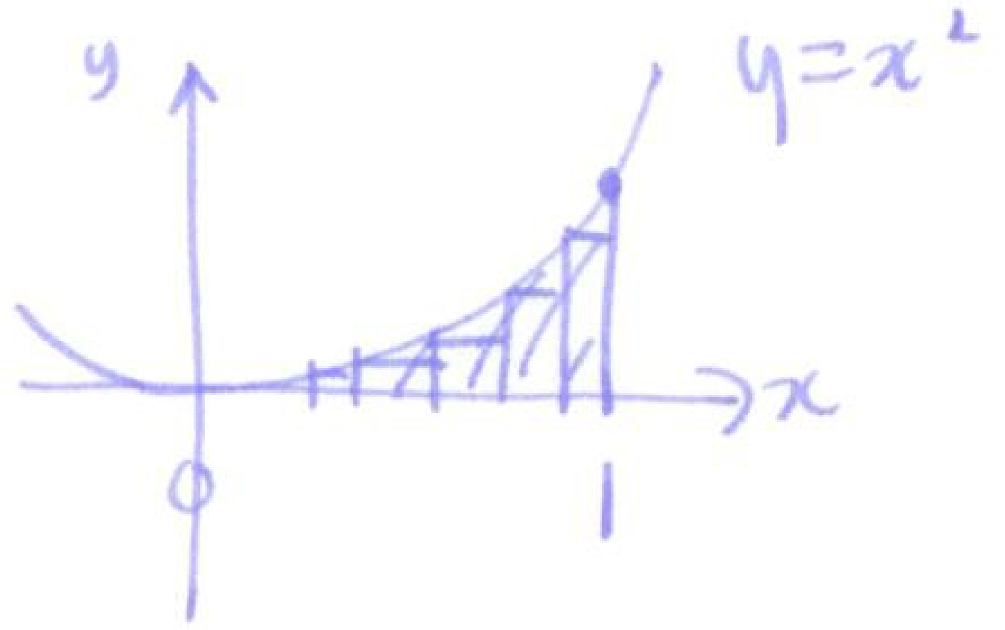
(right endpoint) rectangle approx $R_N = \Delta x \sum_{j=1}^{N} f(a + j\Delta x)$

(midpoint) rectangle approx $M_N = \Delta x \sum_{j=1}^{N} f\left(a + \left(j - \tfrac{1}{2}\right)\Delta x\right)$

useful fact: Thm If $f(x)$ is cts on $[a,b]$ then all three approximations give you the area under the curve:

$$\text{area} = \lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N.$$

Example



n rectangles:

$$L_n = \sum_{k=0}^{n-1} \frac{1}{n} f\left(\frac{k}{n}\right) = \sum_{k=0}^{n-1} \frac{k^2}{n^3}.$$

$$= \frac{1}{n^3} \sum_{k=0}^{n-1} k^2.$$

note: $1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{1}{6} N(N+1)(2N+1)$

proof (induction) $\square$.

$$= \frac{1}{n^3}\frac{1}{6}(n-1)(n)\big(2(n-1)+1\big) = \frac{1}{n^2}\frac{1}{6}(n-1)(2n-1)$$

$$= \frac{1}{6}\,\frac{2n^2-3n+1}{n^2} = \frac{1}{3}\left(1 - \frac{1}{2n} + \frac{1}{6n^2}\right)$$

so $\displaystyle \text{area} = \lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}.$